(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: HOMEWORK 5

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Exceptionally, this homework is 2 pages long: there is stuff on the verso.

Exercise 1

Let $(M, \cdot, e)$ be a monoid. The *writer monad* for M on **Set** is defined by:

- The endofunctor $W_M = (-) \times M$, i.e. $\begin{aligned} A &\mapsto A \times M \\ f &\mapsto f \times \mathrm{id}_M \end{aligned}$
- The unit and multiplication

$$\begin{aligned} \eta_A \colon A &\to A \times M \\ a &\mapsto (a, e) \end{aligned} \qquad \begin{aligned} \mu_A \colon (A \times M) \times M &\to A \times M \\ ((a, m_1), m_2) &\mapsto (a,\ m_1 \cdot m_2) \end{aligned}$$

We admit that η_A and μ_A are natural in A, and that the unit laws hold.

1. Check that the associativity law for μ holds.

2. Describe the composition of $f \colon A \to W_M(B)$ with $g \colon B \to W_M(C)$ in the Kleisli category $\mathbf{Set}_{W_M}$.

3. We now take $(M, \cdot, e)$ to be the free monoid $(\mathbb{N}^*, \cdot, [])$. Let $\texttt{print} \in \mathbf{Set}_{W_{\mathbb{N}^*}}(\mathbb{N}, \mathbb{N})$ – i.e. $\texttt{print} \colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}^*$ – be defined by $\texttt{print}(n) = (n, [n])$. Describe the result of the following composition in $\mathbf{Set}_{W_{\mathbb{N}^*}}$:

$$\{*\} \xrightarrow{F(* \mapsto 6)} \mathbb{N} \xrightarrow{\texttt{print}} \mathbb{N} \xrightarrow{F(n \mapsto n \times 7)} \mathbb{N} \xrightarrow{\texttt{print}} \mathbb{N}$$

where the functor $F \colon \mathbf{Set} \to \mathbf{Set}_{W_{\mathbb{N}^*}}$ is the left part of the adjunction induced by the monad $W_{\mathbb{N}^*}$. (No detailed justification necessary.)

Exercise 2

Let's look at adjunctions between preorders (also called *Galois correspondences*):

preordered set	category (small, with $\lvert\mathcal{C}(A, B)\rvert \leqslant 1$)
monotone function	functor
[topic of this exercise]	adjunctions

They are important for instance in abstract interpretation (which is used for static analysis of programs). Two monotone functions $\ell \colon X \to Y$ and $r \colon Y \to X$ are **adjoint** – notation: $\ell \dashv r$ – when, writing $\leqslant$ for the preorders both on X and on Y,

$$\forall x \in X,\ \forall y \in Y,\ \ell(x) \leqslant y \iff x \leqslant r(y) \qquad [\textit{“hom-set isomorphism” point of view}]$$

For example, there is an adjunction $\iota \dashv \lfloor - \rfloor$ between the inclusion map $\iota \colon \mathbb{Z} \to \mathbb{R}$ ($\iota(n) = n$) and the "floor" function which rounds down a number ($\lfloor 3.14 \rfloor = 3$).

Questions 3 and 4 are just special cases of theorems on adjoint functors from the lectures, but you should provide proofs "from scratch" for preorders.

1. We just described the right adjoint to ι, but it also has a left adjoint – what is it?

2. Let $f \colon A \to B$ be a map between two sets. The associated inverse image map $f^{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A)$ and direct image map $\exists_f \colon X \in \mathcal{P}(A) \mapsto f(X) \in \mathcal{P}(B)$ are both monotone with respect to inclusion. Show that $\exists_f \dashv f^{-1}$.

3. [*"unit+counit" point of view*] In general, show that $\ell \dashv r$ if and only if

$$\forall x \in X,\ x \leqslant r(\ell(x)) \qquad \text{and} \qquad \forall y \in Y,\ \ell(r(y)) \leqslant y$$

4. [*preservation of products*] Recall that y is said to be an **infimum** of $S \subseteq Y$ when

$$\forall z \in Y,\ [z \leqslant y \iff (\forall s \in S,\ z \leqslant s)]$$

Through the preorder/category correspondence, infima correspond to products. For example, $\bigcap S = \bigcap_{X \in S} X$ is the unique[1] infimum in $(\mathcal{P}(\mathcal{A}), \subseteq)$ of $S \subseteq \mathcal{P}(A)$.

(a) Show that if $r\colon Y \to X$ is a right adjoint (i.e. $\ell \dashv r$ for some ℓ) then r **preserves infima**: if y is an infimum of S then $r(y)$ is an infimum of $\{r(s) \mid s \in S\}$.
(b) What does this property concretely mean concerning f^{-1} (cf. question 2)?
(c) What does the dual property concretely mean concerning $\exists_f$?

5. [*"baby adjoint functor theorem"*] Let $r\colon Y \to X$ be a monotone function. Suppose also that r preserves infima and that every subset of Y admits some infimum. Let

$$\ell\colon x \in X \mapsto \text{some choice}^2 \text{ of infimum of } \{y \in Y \mid x \leqslant r(y)\}$$

Using the characterization of Question 3, show that $\ell \dashv r$.

The questions stop here!

... but here are some examples of the "baby adjoint functor theorem":

- Let M be a monoid. Recall that a **submonoid** of M is a subset containing the unit element and closed under the binary operation. An intersection of submonoids is always a submonoid, and the inclusion map

  $$\iota'\colon \mathrm{Submonoids}(M) \to \mathcal{P}(M)$$

  preserves intersections, so it has an left adjoint $\langle - \rangle \dashv \iota'$. According to Question 5, we have the explicit formula

  $$\begin{aligned} \langle - \rangle\colon \mathcal{P}(M) &\to \mathrm{Submonoids}(M) \\ X &\mapsto \langle X \rangle = \bigcap \{N \text{ submonoid of } M \mid X \subseteq N\} \end{aligned}$$

  Thus, $\langle X \rangle$ is precisely *the submonoid of M generated by X*.
- In $(\mathcal{P}(A), \subseteq)$, unions are suprema (dual to infima). For $f\colon A \to B$, the dual version of the baby adjoint functor theorem tells us that, since

  $$f^{-1}\left(\bigcup S\right) = \bigcup_{X \in S} f^{-1}(X)$$

  f^{-1} must have a right adjoint: $f^{-1} \dashv \forall_f$. Explicitly, we have (this is not quite the formula of Question 5, but it's related):

  $$\begin{aligned} \forall_f(X) &= \{b \in B \mid \forall a \in f^{-1}(\{b\}) : a \in X\} \\ \exists_f(X) &= \{b \in B \mid \exists a \in f^{-1}(\{b\}) : a \in X\} = f(X) \end{aligned}$$

[1] It is unique because $\subseteq$ is a partial order, but there are preorders in which infima are not unique.
[2] Non-uniqueness (see previous footnote) is why we must make a choice.