(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: HOMEWORK 4

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Exercise 1

Let $(\mathcal{C}, \times, T)$ be a cartesian category. The goal of this exercise is to look at an example of reasoning using the pairing $\langle f, g\rangle \colon A \to B \times C$ of two morphisms $f \colon A \to B$ and $g \colon A \to C$, and the equations $\pi_1^{B,C} \circ \langle f, g\rangle = f$ and $\pi_2^{B,C} \circ \langle f, g\rangle = g$ relating it to the projections.

1. Let $f_1 \colon A_1 \to B_1$ and $f_2 \colon A_2 \to B_2$. Write down the definition of the functorial image $f_1 \times f_2 \colon A_1 \times A_2 \to B_1 \times B_2$, defined in Lecture 5, in terms of the pairing operation $\langle -, -\rangle$ and projections.

2. Let $X, Y \in \mathrm{ob}(\mathcal{C})$. Recall that since $(X \times Y, \pi_1^{X,Y}, \pi_2^{X,Y})$ and $(Y \times X, \pi_2^{Y,X}, \pi_1^{Y,X})$ are both products of X and Y, there is a unique morphism $\sigma^{X,Y} \colon X \times Y \to Y \times X$ such that $\pi_2^{Y,X} \circ \sigma^{X,Y} = \pi_1^{X,Y}$ and $\pi_1^{Y,X} \circ \sigma^{X,Y} = \pi_2^{X,Y}$, and it is an iso. (This is the "uniqueness up to unique iso" of universal properties, in the case of products.)

Write down the definition of $\sigma_{A,B}$ using pairings and projections, and check that it satisfies the above equations.

3. Prove that $\sigma^{B_2,B_1} \circ (g \times f) \circ \sigma^{A_1,A_2} = f \times g$.

Exercise 2

Consider the following functor – it is the composition of the forgetful functor $\mathbf{Ord} \to \mathbf{Set}$ with the functor F introduced in Homework 2:

$$G \colon \mathbf{PreOrd}^{\mathrm{op}} \to \mathbf{Set}$$
$$(A, \preceq_A) \in \mathrm{ob}(\mathbf{PreOrd}^{\mathrm{op}}) \mapsto \mathrm{UpClosed}(A, \preceq_A)$$
$$f \in \mathbf{PreOrd}^{\mathrm{op}}(A, B) \mapsto (X \in \mathrm{UpClosed}(A, \preceq_A) \mapsto f^{-1}(X))$$

(Reminder: $\mathrm{UpClosed}(A, \preceq_A)$ consists of subsets $X \subseteq A$ which are upwards closed, i.e. $\forall x \in X,\ \forall a \in A,\ x \preceq a \Rightarrow a \in X$.)

Let $\Omega = \{0, 1\}$ equipped with the partial order such that $0 \leqslant 1$.

1. Show that G is representable, using Ω as part of the representation.

2. For any set X, let $H(X) = \Omega^X$ equipped with the product order (which was used in Lecture 4 to explicitly construct categorical products of arbitrary families in $\mathbf{Ord}$ and $\mathbf{PreOrd}$). Explain how to extend H to a functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{PreOrd}$. (Say how H acts on morphisms; you need not check the functor axioms.)

3. Prove that we have an adjunction $H^{\mathrm{op}} \dashv G$, using the "natural isomorphism between homsets" definition.

4. Describe the unit and counit of this adjunction.