(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: HOMEWORK 3

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Exercise 1

We write $\mathcal{P}(A) = \{X \mid X \subseteq A\}$ for the powerset of the set A. For two sets A and B, a subset $X \in \mathcal{P}(A)$ and a binary relation $R \subseteq A \times B$, let

$$R_\dagger X = \{b \in B \mid \exists a \in X : (a, b) \in R\}$$

Consider the following endofunctor on the category $\mathbf{Rel}$ of relations (we admit that it is a functor, i.e. preserves composition and identities):

$$F\colon \mathbf{Rel} \to \mathbf{Rel}$$
$$A \in \mathrm{ob}(\mathbf{Rel}) \mapsto \mathcal{P}(A)$$
$$R \in \mathbf{Rel}(A, B) \mapsto \{(X, R_\dagger X) \mid X \in \mathcal{P}(A)\}$$

1. Show that the following family $(\rho_A)_{A \in \mathrm{ob}(\mathbf{Rel})}$ of binary relations defines a natural transformation $\rho\colon F \Rightarrow \mathrm{Id}_{\mathbf{Rel}}$: $\rho_A = \{(X, x) \mid X \in \mathcal{P}(A),\ x \in X\}$.

2. Show that the following family $(\sigma_A)_{A \in \mathrm{ob}(\mathbf{Rel})}$ of relations *does not* define a natural transformation from F to $\mathrm{Id}_{\mathbf{Rel}}$: $\sigma_A = \{(\{a\}, a) \mid a \in A\}$.
Hint: look at the naturality condition for $\{(0, 1), (0, 2)\} \in \mathbf{Rel}(\{0\}, \{1, 2\})$.

3. Prove that the functor F is faithful. Is it full?

Exercise 2

A *partial function* between sets is a function that might be defined only on a subset of its domain, e.g. the inverse function $\mathrm{inv}\colon x \mapsto 1/x$ on real numbers is defined on $\mathbb{R} \setminus \{0\}$. We write $\mathrm{inv}\colon \mathbb{R} \rightharpoonup \mathbb{R}$ to say that inv is a partial function from $\mathbb{R}$ to $\mathbb{R}$ (note the special shape of the arrow $\rightharpoonup$). The category $\mathbf{PSet}$ has all sets as objects; $\mathbf{PSet}(A, B) = \{f \mid f\colon A \rightharpoonup B\}$ for two sets A and B; composition is defined as expected and the identities are identity functions.

Consider also the following category $\mathcal{C}$, where composition is the usual function composition and identities are (again) identity functions:

- $\mathrm{ob}(\mathcal{C}) = \{(A, a) \mid A \text{ is a set and } a \in A\}$
- $\mathcal{C}((A, a), (B, b)) = \{f \mid f\colon A \to B \text{ (usual map between sets)},\ f(a) = b\}$

Show that the operations on objects[1] $F(A) = (\{\varnothing\} \cup \{\{a\} \mid a \in A\},\ \varnothing)$ and $G((A, a)) = A \setminus \{a\}$ can be extended to *functors* $F\colon \mathbf{PSet} \to \mathcal{C}$ and $G\colon \mathcal{C} \to \mathbf{PSet}$ that form an *equivalence of categories* between $\mathbf{PSet}$ and $\mathcal{C}$.

[1]The variant $F'(A) = (\{\mathsf{Some}(a) \mid a \in A\} \cup \{\mathsf{None}\}, \mathsf{None})$ based on the "option data type" could have been used instead to construct an equivalence of categories. What's important is to use, for the left side of the pair, a *coproduct* in $\mathbf{Set}$ of A with some singleton.