(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: HOMEWORK 1

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Exercise 1

Let $\mathcal{C}$ be the full subcategory of $\mathbf{Set}$ whose objects are $\varnothing$ and $\{4\}$.

1. What are the morphisms of this category? (Recall that for any set X, there is a single relation between $\varnothing$ and X, and this relation is a function.)

2. Exhibit a functor $F\colon C \to \mathbf{Set}$ such that:

- $F(\varnothing) = F(\{4\}) = \mathbb{N}$,
- there exists some morphism f of $\mathcal{C}$ such that $F(f)$ is the function $n \mapsto n+1$.

3. Show that there does *not* exist any morphism g such that $F(g) = F(f) \circ F(f)$. In particular, why *cannot* we say that "$F(f \circ f) = F(f) \circ F(f)$" even though F is a functor?

Exercise 2

Throughout this exercise, we fix some monoid M. Let I be the subset of M that consists of the invertible elements. We admit that I is a submonoid of M, and that it is a group. Let $i\colon I \to M$ be the inclusion map defined by $\iota(y) = y$.

1. Show that the following universal property holds: for every group G and monoid homomorphism $h\colon G \to M$, there exists a unique homomorphism $h'\colon G \to I$ such that $h = i \circ h'$. (Hint: what does that mean concerning each $h(x)$ for $x \in G$?)

2. Let J be a group and $j\colon J \to M$ be a homomorphism. Suppose that the above universal property holds for (J, j) instead of (I, i): for every group G and monoid homomorphism $h\colon G \to M$, there exists a unique homomorphism $h'\colon G \to J$ such that $h = j \circ h'$. Prove that I and J are isomorphic.

(You should be able to do that using only the universal property, without using the "concrete" definition of I.)

3. Can you express this as a universal morphism from F to M for some functor F seen in the lectures?