(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: LECTURE 10

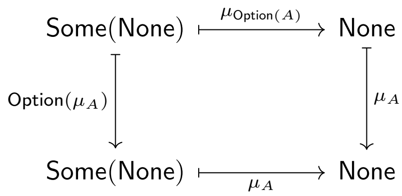
14 OCTOBER 2024 — L. T. D. NGUYỄN

Last time: Monads induced by adjunctions = internal monoids in categories of endofunctors (and connections with side effects in programming).

Note that the "internal monoid" definition is useful to check that a monad is a monad without having to guess an adjunction: it suffices to verify that μ and η satisfy the associativity and unit laws. For instance, we may turn the "option data type" into a monad over $\mathbf{Set}$ by defining directly μ and η as follows:

$$\begin{aligned} \mu_A \colon \mathsf{Option}(\mathsf{Option}(A)) &\to \mathsf{Option}(A) \\ \mathsf{Some}(\mathsf{Some}(a)) &\mapsto \mathsf{Some}(a) \\ \mathsf{Some}(\mathsf{None}) &\mapsto \mathsf{None} \\ \mathsf{None} &\mapsto \mathsf{None} \end{aligned} \qquad \begin{aligned} \eta_A \colon A &\to \mathsf{Option}(A) \\ a &\mapsto \mathsf{Some}(a) \end{aligned}$$

To show that it is indeed a monad, it suffices to check the associativity and unit laws. This can be done manually, for instance one of the cases for associativity is:

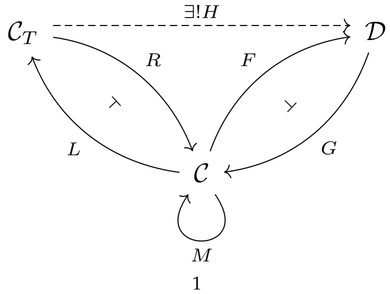


(the other cases for associativity start with the values None, $\mathsf{Some}(\mathsf{Some}(\mathsf{None}))$ or $\mathsf{Some}(\mathsf{Some}(\mathsf{Some}(a)))$ in the upper left corner, and then there remains to check the unit laws; let us skip all this).

The option monad models the computational effect of *partiality*, in other words, of the possibility of *failure*. A Kleisli morphism $f \in \mathbf{Set}_{\mathsf{Option}}(X, Y)$, i.e. a function $X \to \mathsf{Option}(Y)$, can indeed be seen as a *partial* function $X \rightharpoonup Y$ — we shall come back to this point. (A partial function is a function that is possibly undefined on part of its domain, such as $(x \mapsto 1/x) \colon \mathbb{R} \rightharpoonup \mathbb{R}$.)

To conclude our study of monads let us mention that, among the adjunctions that induce a monad, the Kleisli adjunction is particularly canonical.

Theorem. *Let $F \dashv G$ be an adjunction with $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$. Let (M, μ, η) be the induced monad (so $M = G \circ F$). Let us write $L \colon \mathcal{C} \to \mathcal{C}_M$ and $R \colon \mathcal{C}_M \to \mathcal{C}$ be the pair of adjoint functors defined at the end of the last lecture. There exists a unique functor $H \colon \mathcal{C}_M \to \mathcal{D}$ such that $H \circ L = F$ and $R = G \circ H$.*



(We will not prove this theorem in this course.)

Remark. This is an *initiality* property of the Kleisli adjunction: it says that it is an initial object in a category whose objects are adjunctions inducing the monad M, and whose morphisms are functors subject to certain conditions.

Comonads

This notion is dual to monads:

Definition. A comonad on $\mathcal{C}$ is a monad on $\mathcal{C}^{\mathrm{op}}$.

Explicitly, a comonad (W, ν, ε) on $\mathcal{C}$ consists of an endofunctor $W\colon \mathcal{C} \to \mathcal{C}$ with two natural transformations $\nu\colon W \Rightarrow W \circ W$ (comultiplication) and $\varepsilon\colon W \Rightarrow \mathrm{Id}_{\mathcal{C}}$ (counit) such that the following equivalent conditions hold:

- $W = F \circ G$ and $\nu = F(\eta_G)$ for some adjunction $F \dashv G$ with unit η and counit ε — the comonad is said to be induced by this adjunction;
- dual versions of the associativity and unitality laws hold: comonads are "internal comonoids in $[\mathcal{C}, \mathcal{C}]$" (internal comonoids will come up again in Olivier Laurent's half of the course).

Proposition. *These two conditions are indeed equivalent. In fact, any comonad (W, ν, ε) on $\mathcal{C}$ induced by a "coKleisli adjunction"; the coKleisli category $\mathcal{C}_W$ has the same objects as $\mathcal{C}$, and $\mathcal{C}_W(A, B) = \mathcal{C}(W(A), B)$.*

Proof. By applying the theorems on monads from the previous lecture to $\mathcal{C}^{\mathrm{op}}$. □

Let us give an example of a comonad. Let $(\mathcal{C}, \&, \top)$ be a cartesian category and $A \in \mathrm{ob}(\mathcal{C})$. Let $W = (- \mathbin{\&} A)$ (partial application of the product bifunctor) and

$$\nu_X = \langle \mathrm{id}_{X \& A}, \pi_2^{X,A} \rangle \in \mathcal{C}(X \mathbin{\&} A,\ (X \mathbin{\&} A) \mathbin{\&} A) \qquad \varepsilon_X = \pi_1^{X,A} \in \mathcal{C}(X \mathbin{\&} A,\ X)$$

Both ν_X and ε_X are natural in X, and one can check that they define a comonad structure on $(- \mathbin{\&} A)$.

(Remark: the dual of this example is the monad $-\oplus A$; for $A = \{\mathsf{None}\} \in \mathrm{ob}(\mathbf{Set})$ we get something naturally isomorphic to the Option monad!)

Isomorphism and equivalence of categories

We mentioned earlier that morphisms in $\mathbf{Set}_{\mathsf{Option}}$ are "the same thing" as partial functions. This can be made formal by saying that $\mathbf{Set}_{\mathsf{Option}}$ is *isomorphic* to the category $\mathbf{PSet}$ of sets and partial functions.

Definition. An *isomorphism* of categories is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that, for some $G\colon \mathcal{D} \to \mathcal{C}$, we have $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}}$.

The following defines an isomorphism of categories:

$$\begin{aligned} \mathbf{Set}_{\mathsf{Option}} &\to \mathbf{PSet} \\ X \text{ set} &\mapsto X \\ f \in \mathbf{Set}_{\mathsf{Option}}(X, Y) &\mapsto \begin{bmatrix} X \rightharpoonup Y \\ x \mapsto \begin{cases} y & \text{if } f(x) = \mathsf{Some}(y) \\ \text{undefined} & \text{if } f(x) = \mathsf{None} \end{cases} \end{bmatrix} \end{aligned}$$

(the fact that it is a functor, and the definition of the inverse, are left as exercises). Another example of isomorphism of categories is the self-duality of $\mathbf{Rel}$:

$$\begin{aligned} \mathrm{Reverse}\colon \mathbf{Rel} &\to \mathbf{Rel}^{\mathrm{op}} \qquad \text{(with inverse } \mathrm{Reverse}^{\mathrm{op}}) \\ X &\mapsto X \\ R \subseteq X \times Y &\mapsto \{(y, x) \mid (x, y) \in R\} \end{aligned}$$

Note that in both these examples, the functors are "identity on objects". When a construction that purports to show that two categories are somehow "the same" changes the objects, it often turns out that the notion of isomorphism of categories is slightly too restrictive. Instead, *equivalences of categories* are more frequent.

Definition. An *equivalence* of categories between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$.

The difference with isomorphisms of categories is that equalities of functors have been relaxed to natural isomorphisms.

For example, consider the category $\mathbf{Set}_{\mathrm{pt}}$ of *pointed sets*:

objects: pairs (X, x) where X is a set and $x \in X$

morphisms from (X, x) to (Y, y): functions $f\colon X \to Y$ such that $f(x) = y$

The idea is that the distinguished element $x \in X$ can be seen as the "error value" in X. The following is an equivalence of categories:

$$
\begin{aligned}
F\colon \mathbf{PSet} &\to \mathbf{Set}_{\mathrm{pt}} \\
X &\mapsto (\mathsf{Option}(X), \mathsf{None}) \\
(f\colon X \rightharpoonup Y) &\mapsto \begin{bmatrix} \mathsf{Some}(x) \mapsto \begin{cases} \mathsf{Some}(y) & \text{if } f(x) = y \\ \mathsf{None} & \text{if } f(x) \text{ undefined} \end{cases} \\ \mathsf{None} \mapsto \mathsf{None} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
G\colon \mathbf{Set}_{\mathrm{pt}} &\to \mathbf{PSet} \\
(X, x) &\mapsto X \setminus \{x\} \\
f \in \mathbf{Set}_{\mathrm{pt}}((X, x), (Y, y)) &\mapsto \begin{bmatrix} X \setminus \{x\} \rightharpoonup Y \setminus \{y\} \\ x' \mapsto \begin{cases} \text{undefined} & \text{if } f(x') = y \\ f(x') & \text{otherwise} \end{cases} \end{bmatrix}
\end{aligned}
$$

Note that $G(F(X)) = \{\mathsf{Some}(x) \mid x \in X\}$ is not *equal* to X, only *in bijection* with it; and this bijection is natural in X, that is, $G \circ F \cong \mathrm{Id}_{\mathbf{PSet}}$. (Beware: while ordinary bijections are indeed isomorphisms in $\mathbf{PSet}$, naturality should be checked with respect to partial functions!) Similarly, $F \circ G \cong \mathrm{Id}_{\mathbf{Set}_{\mathrm{pt}}}$.

Other examples of equivalences of categories include:

$\mathbf{Mon}$ is equivalent to $\mathbf{1obCat}$: that is, to the category of small categories with only 1 object. We have seen previously that $M \in \mathrm{ob}(\mathbf{Mon}) \mapsto \mathcal{C}_M$ makes homomorphisms in $\mathbf{Mon}(M, N)$ correspond to functors $\mathcal{C}_M \to \mathcal{C}_N$, so it defines a functor $F\colon \mathbf{Mon} \to \mathbf{1obCat}$. In the converse direction, $G\colon \mathbf{1obCat} \to \mathbf{Mon}$ sends a category with one object to the monoid of endomorphisms of its unique object. We have $G \circ F = \mathrm{Id}_{\mathbf{Mon}}$. However, $F(G(\mathcal{C})) =$ a category with the same morphisms as $\mathcal{C}$, but with its unique object "renamed" into the constant $*$ used by the $M \mapsto \mathcal{C}_M$ construction. Thus $F(G(\mathcal{C}))$ is not equal to $\mathcal{C}$, but they are isomorphic, naturally in $\mathcal{C}$.

$\mathbf{PreOrd}$ is equivalent to the category of small thin categories: the same idea as above, where a thin category is defined to be a category where for any two objects, there is at most one morphism between them.

$\mathbf{Set}^I$ vs $\mathbf{Set}/I$: Let I be a set. $\mathbf{Set}/I$ is the category:
- whose objects are pairs (A, f) with $f\colon A \to I$;
- whose morphisms from (A, f) to (B, g) are functions $h\colon A \to B$ such that $f = g \circ h$ (which can be pictured as a commuting triangle).

(This is a variant of the "$F \downarrow X$" from Lecture 5.)

Let is sketch an equivalence between $\mathbf{Set}/I$ and the product category $\mathbf{Set}^I$. The functor $\mathbf{Set}/I \to \mathbf{Set}^I$ sends $(A,\ f\colon A \to I)$ to $(f^{-1}(i))_{i \in I}$;

in the other direction, $(A_i)_{i\in I}$ is sent to the dependent sum $\sum_{i\in I} A_i$ with the map $(i, a) \mapsto i$. (The actions on morphisms are the obvious ones.) Their composition turns $(A_i)_{i\in I}$ into $(\{i\}\times A_i)_{i\in I}$ which is indeed naturally isomorphic to, but not equal to, what we started with.

This is a basic example of the correspondences between "fibered" and "indexed" points of view. Such correspondences play an important role in the semantics of polymorphic types and dependent types.

The terminal category (with only 1 object and its identity morphism): it is equivalent to any category with *exactly* one morphism between any two objects, i.e. any thin category that corresponds to a trivial preorder; the number of objects doesn't matter (it may be uncountably infinite...).

Proposition. *Equivalences of categories are closed under composition.*

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$, $F'\colon \mathcal{D} \to \mathcal{E}$ and $G'\colon \mathcal{E} \to \mathcal{D}$. Let $\alpha\colon G \circ F \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and $\beta\colon G' \circ F' \Rightarrow \mathrm{Id}_{\mathcal{D}}$. Then $\alpha \circ G(\beta_F)\colon G \circ G' \circ F' \circ F \Rightarrow \mathrm{Id}_{\mathcal{C}}$; and if α and β are natural isomorphisms, then so is $\alpha \circ G(\beta_F)$. Symmetrically, from $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$ and $F' \circ G' \cong \mathrm{Id}_{\mathcal{E}}$ one can get $F' \circ F \circ G \circ G' \cong \mathrm{Id}_{\mathcal{E}}$. □

Next, we state without proof some key properties of equivalences of categories.

Proposition. *The following are equivalent:*

- *$F\colon \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories;*
- *the functor F has all three properties below:*
  - **full:** *for all $A, B \in \mathrm{ob}(\mathcal{C})$, the map $f \in \mathcal{C}(A, B) \mapsto F(f) \in \mathcal{D}(F(A), F(B))$ is surjective;*
  - **faithful:** *for all A, B, the map $f \in \mathcal{C}(A, B) \mapsto F(f) \in \mathcal{D}(F(A), F(B))$ is injective;*
  - **essentially surjective:** *for every $X \in \mathrm{ob}(\mathcal{D})$ there exists $A \in \mathrm{ob}(\mathcal{C})$ such that $F(A) \cong X$.*

The top-to-bottom implication is not difficult. The converse however uses the axiom of choice!

Theorem (Adjoint equivalences)**.** *If F and G form an equivalence of categories, then there are adjunctions $F \dashv G$ and $G \dashv F$.*

More precisely, one can find natural isomorphisms $\eta\colon \mathrm{Id} \Rightarrow G \circ F$ *and* $\varepsilon\colon F \circ G \Rightarrow \mathrm{Id}$ *that are the unit and counit for an adjunction $F \dashv G$ (they may be different from the isos originally witnessing that F and G form an equivalence of categories). Automatically, then, ε^{-1} and η^{-1} are respectively the unit and counit for an adjunction $G \dashv F$.*

Corollary. *If F is part of an equivalence of categories, it preserves both products and coproducts.*

Proof. We have seen that right adjoints preserve products, and dually left adjoints preserve coproducts. According to the above theorem, F is both a left adjoint and a right adjoint. □

This is a general phenomenon: useful structure in category theory tends to be preserved by equivalences. Thus, equivalences are a well-behaved notion of being "essentially the same category".

Concluding cultural remarks: Yoneda, presheaves, and (co)limits

In our study of representable functors, we saw that a representation (A, θ), where $\theta\colon \mathcal{C}(A, -) \Rightarrow \mathcal{D}(X, F(-))$ is a natural bijection, is entirely characterised by a universal morphism (A, φ) with $\varphi = \theta_A(\mathrm{id}_A)$. The moral of the story is that "a big natural iso is fully characterised by a small piece of data".

According to the *Yoneda lemma*, this works for natural transformations in general:

Lemma (Yoneda)**.** *$(\theta\colon \mathcal{C}(A,-) \Rightarrow F) \mapsto \theta_A(\mathrm{id}_A) \in F(A)$ is a bijection.*

Proof idea. By naturality, one can show that $\theta_B(f) = F(f)(\theta_A(\mathrm{id}_A))$ for $f \in \mathcal{C}(A,B)$ — the idea is the same as in Lecture 7 concerning representable functors! This shows injectivity. For surjectivity one can check that $f \in \mathcal{C}(A,B) \mapsto F(f)(x) \in F(B)$ is natural in B for any $x \in F(A)$. □

Corollary (Yoneda embedding)**.** $\begin{array}{l}\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \\ X \mapsto \mathcal{C}(-,X)\end{array}$ *is a full and faithful functor.*

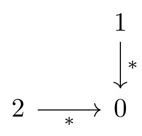
Idea. The Yoneda lemma applied to $\mathcal{C}^{\mathrm{op}}$ gives us, in particular, a natural bijection between the natural transformations $\mathcal{C}(-,X) \Rightarrow \mathcal{C}(-,Y)$ and the morphisms in $\mathcal{C}(X,Y)$. □

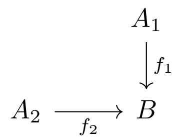
The functor category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is also called the category of *presheaves* (in French: *préfaisceaux*) over $\mathcal{C}$; it is a sort of "completion" of $\mathcal{C}$ with nice properties. In fact it is $\mathcal{C}$ with all *colimits* freely added. Which is a good pretext to talk about limits and colimits!

Let $\mathcal{J}$ and $\mathcal{C}$ be two categories. The diagonal functor $\Delta\colon \mathcal{C} \to [\mathcal{J}, \mathcal{C}]$ maps every object $X \in \mathrm{ob}(\mathcal{C})$ to the constant functor $\mathcal{J} \to \mathcal{C}$ (i.e. object of $[\mathcal{J}, \mathcal{C}]$) equal to X – $\Delta(X)(Y) = X$ and $\Delta(X)(f) = \mathrm{id}_X$ – and every morphism to a constant natural transformation – $\Delta(g)_Y = g$ for $g \in \mathcal{C}(X, X')$.

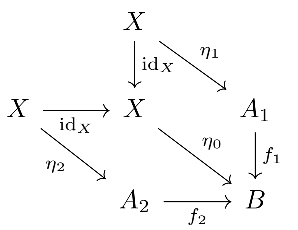
Definition. A *limit* in $\mathcal{C}$ is a universal morphism from the above diagonal functor $\Delta\colon \mathcal{C} \to [\mathcal{J}, \mathcal{C}]$ to some object $F \in [\mathcal{J}, \mathcal{C}]$, for some category $\mathcal{J}$.

What does this mean? Let's take $\mathcal{J}$ to be a very simple category, with only 3 objects and 2 non-identity morphisms, drawn below (left). A functor $F\colon \mathcal{J} \to \mathcal{C}$ corresponds to choosing 3 objects and 2 morphisms in $\mathcal{C}$, as follows (right):

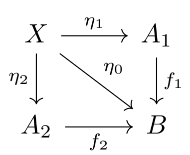




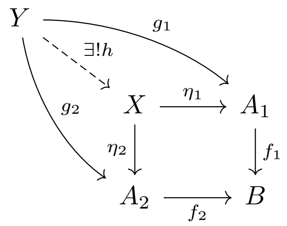
$\Delta(X)$ corresponds to the case where the 3 chosen objects are equal to X, and the 2 chosen morphisms are id_X. So a natural transformation $\eta\colon \Delta(X) \Rightarrow F$ is a family $(\eta_X)_{X \in \{0,1,2\}}$ making the diagram below commute:



or equivalently



The morphism η_0 is kind of redundant, it's determined by $\eta_0 = f_1 \circ \eta_1 = f_2 \circ \eta_2$, so we can omit it in diagrams. If (X, η) is a universal morphism from Δ to F, its universal property is given by the commutative diagram below; the quantifiers in the statement are "for every (Y, g), there exists a unique h":



This special case of limit is called a *pullback*: (X, η_1, η_2) is the pullback of (f_1, f_2). Two examples:

- In **Set**, if A_1 and A_2 are two subsets of B, and f_1, f_2 are their inclusion maps, then the intersection $A_1 \cap A_2$ with two inclusion maps is a pullback of f_1 and f_2.
- If B is a terminal object and f_i is the unique morphism from A_i to the terminal object, then a pullback is the same as a product of A_1 and A_2.

Pullbacks are the limits whose "shape" $\mathcal{J}$ is a specific category with 3 objects. Other kinds of limits can be obtained by varying $\mathcal{J}$. Dually, *colimits* in $\mathcal{C}$ are limits in $\mathcal{C}^{\mathrm{op}}$, i.e. universal morphisms from some F to $\Delta\colon \mathcal{C} \to [\mathcal{C}, \mathcal{J}]$.

Theorem. *Right adjoints preserve limits, and left adjoints preserve colimits.*

(This indeed generalizes what we saw before, since a product is the same as a limit whose shape $\mathcal{J}$ is a category where the only morphisms are identities.)