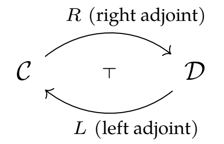
(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: LECTURE 8

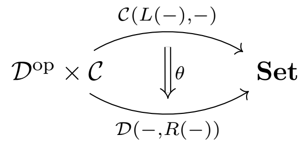
7 OCTOBER 2024 — L. T. D. NGUYỄN

Last time: we defined an *adjunction* $L \dashv R$, also denoted by



when one of the three equivalent conditions below is satisfied:

- (when $\mathcal{C}$ and $\mathcal{D}$ are locally small) there is a natural *isomorphism*



  — to be explicit, the naturality square is:

$$\begin{array}{ccc} \mathcal{C}(L(X), A) & \xrightarrow{\theta_{X,A}} & \mathcal{D}(X, R(A)) \\ {\scriptstyle g\circ(-)\circ L(f)}\downarrow & & \downarrow{\scriptstyle R(g)\circ(-)\circ f} \\ \mathcal{C}(L(Y), B) & \xrightarrow[\theta_{Y,B}]{} & \mathcal{D}(Y, R(B)) \end{array}$$

- there is a natural transformation $\eta\colon \mathrm{Id}_{\mathcal{D}} \Rightarrow R \circ L$, called the *unit*, such that $(L(X), \eta_X)$ is a universal morphism from X to R for every $X \in \mathrm{ob}(\mathcal{D})$
- there is a natural transformation $\varepsilon\colon L \circ R \Rightarrow \mathrm{Id}_{\mathcal{C}}$, called the *counit*, such that $(R(A), \varepsilon_A)$ is a universal morphism from L to A for every $A \in \mathrm{ob}(\mathcal{C})$

The proof of equivalence went through the presentation of universal morphisms using representable functors.

Remark. We admit that the last 2 conditions are equivalent even when $\mathcal{C}$ and $\mathcal{D}$ are not locally small.

Remark. $L \dashv R \iff R^{\mathrm{op}} \dashv L^{\mathrm{op}}$.

We also saw a bijective correspondence between the *data* $\theta, \eta, \varepsilon$:

$$\eta_X = \theta_{X,L(X)}(\mathrm{id}_{L(X)}) \qquad \theta_{X,A}(f) = R(f) \circ \eta_X$$
$$\varepsilon_A = \theta^{-1}_{R(A),A}(\mathrm{id}_{R(A)}) \qquad \theta^{-1}_{X,A}(g) = \varepsilon_A \circ L(g)$$

The remarkable fact here is that either η or ε suffice to determine θ. The equations themselves can be guessed as "the only things that make sense given the types".

Examples:

Free/forgetful adjunction between Mon and Set: We have already seen that $(-)^* \dashv U$ with the unit and the natural bijection

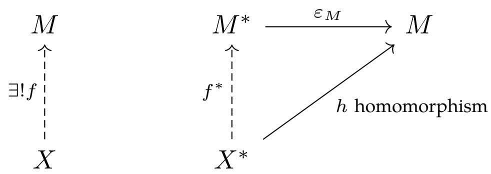
$$\eta_X(x) = [x] \qquad \theta_{X,M}(h) = (x \mapsto h([x]))$$

We can also derive the counit $\varepsilon_M \in \mathbf{Mon}(U(M)^*, M)$ as the only solution to $\mathrm{id}_M = (m \mapsto \varepsilon_M([m]))$, i.e. $\forall m \in M$, $\varepsilon_M([m]) = m$. Thus:

$$\begin{aligned} \varepsilon_M \colon M^* &\to M \\ [m_1, \ldots, m_n] &\mapsto m_1 \cdot \ldots \cdot m_n \end{aligned}$$

This is a variant of the "fold" combinators in functional programming (in Haskell, ε_M is called `mconcat`).

We then know that $(U(M), \varepsilon_M)$ is a universal morphism from $(-)^*$ to M, that is:

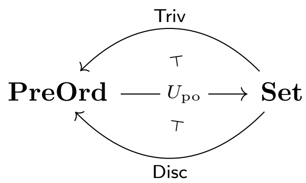


Trivial preorder: Let $U_{\mathrm{po}} \colon \mathbf{PreOrd} \to \mathbf{Set}$ be the forgetful functor.

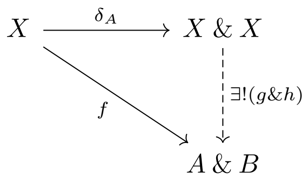
For $X \in \mathrm{ob}(\mathbf{Set})$, let $\mathsf{Triv}(X) = (X, \leqslant_{\mathrm{triv}})$ where $x \leqslant_{\mathrm{triv}} y$ is always true. We have seen (cf. Homework 1) that $(\mathsf{Triv}(X), \mathrm{id}_X)$ is a universal morphism from U_{po} to X. Therefore Triv has a unique extension to a functor $\mathbf{Set} \to \mathbf{PreOrd}$ (as expected, $\mathsf{Triv}(f) = f$) that makes id a natural transformation $U_{\mathrm{po}} \circ \mathsf{Triv} \Rightarrow \mathrm{Id}_{\mathbf{Set}}$, the counit of the adjunction $U_{\mathrm{po}} \dashv \mathsf{Triv}$.

- the natural bijection $\mathbf{Set}(X, Y) \to \mathbf{PreOrd}((X, \leqslant), (Y, \leqslant_{\mathrm{triv}}))$ sends a function to itself — indeed, the domain and codomain are equal!
- the unit is $(\mathrm{id}_X^{\mathbf{Set}})_{(X,\leqslant) \in \mathrm{ob}(\mathbf{PreOrd})} \colon \mathrm{Id}_{\mathbf{PreOrd}} \Rightarrow \mathsf{Triv} \circ U_{\mathrm{po}}$, this means that the identity map is always monotone from $(X, \leqslant)$ to $(X, \leqslant_{\mathrm{triv}})$.

Discrete preorder (i.e. equality): $\mathsf{Disc}(X) = (X, =)$ is *left* adjoint to U_{po} and the unit, counit and natural bijection are composed of identity maps, thanks to the equality $\mathbf{PreOrd}((X, =), (Y, \leqslant)) = \mathbf{Set}(X, Y)$. To sum up:



Products: In a cartesian category, $\Delta \dashv (- \,\&\, -)$ with the counit given by the projections: $(\pi_1, \pi_2) \colon \Delta \circ (- \,\&\, -) \Rightarrow \mathrm{Id}_{\mathcal{C} \times \mathcal{C}}$. The natural bijection $\mathcal{C}(X, A) \times \mathcal{C}(X, B) \cong \mathcal{C}(X,\, A \,\&\, B)$ is the pairing function. Therefore, the unit is $\delta \colon \mathrm{Id}_{\mathcal{C}} \Rightarrow (- \,\&\, -) \circ \Delta$ whose components are *diagonal morphisms* $\delta_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle \in \mathcal{C}(A,\, A \,\&\, A)$; the associated universal property is



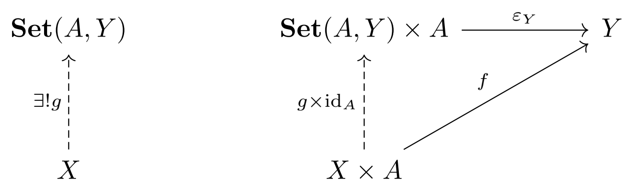
Coproducts: In a cocartesian category, $(- \oplus -) \dashv \Delta$. with the unit (ι_1, ι_2) given by the coprojections. The counit consists of codiagonal morphisms from $A \oplus A$ to A.

Product/function adjunction in Set: We have $\overbrace{(-) \times A}^{\text{partial application of product bifunctor}} \dashv \mathbf{Set}(A, -)$ for any set A. (This example is of major importance: it presents the structure of *cartesian closed category* on $\mathbf{Set}$.)

Let us start with the natural bijection, which is *curryfication*:

$$\begin{aligned} \mathbf{Set}(X \times A,\, Y) &\cong \mathbf{Set}(X,\, \mathbf{Set}(A, Y)) \\ f &\mapsto \big[x \mapsto (a \mapsto f(x,a))\big] \\ \big[(x,a) \mapsto g(x)(a)\big] &\leftarrow\!\shortmid g \end{aligned}$$

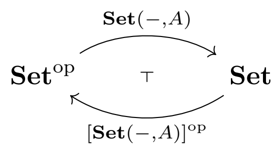
By taking $f = \mathrm{id}_{X \times A}$ we get the unit $\eta_X : x \mapsto (a \mapsto (x,a))$. By taking $g = \mathrm{id}_{\mathbf{Set}(A,Y)}$ we get the counit $\varepsilon_Y : (h,a) \mapsto h(a)$ — this *evaluation map* has the universal property:



Contravariant hom-functors in Set are self-adjoint: We have:

$$\begin{aligned} \mathbf{Set}(X,\, \mathbf{Set}(Y, A)) &\cong \mathbf{Set}(X \times Y,\, A) \\ &\cong \mathbf{Set}(Y \times X,\, A) \\ &\cong \mathbf{Set}(Y,\, \mathbf{Set}(X, A)) = \mathbf{Set}^{\mathrm{op}}(\mathbf{Set}(X, A),\, Y) \end{aligned}$$

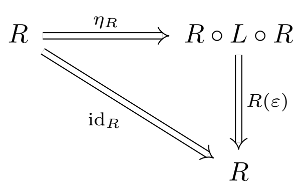
naturally in X and Y. This is a composition of natural isomorphism, where the step in the middle is realized by the isomorphism $\mathbf{Set}(\sigma_{Y,X}, A)$ for $\sigma_{Y,X} \in \mathrm{Iso}_{\mathbf{Set}}(Y \times X,\, X \times Y)$. Therefore:



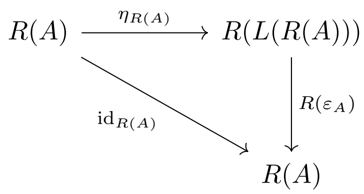
As a *non-example*, $U : \mathbf{Mon} \to \mathbf{Set}$ has no *right* adjoint (even though it has the left adjoint $(-)^*$) — we will be able to give a proof later.

We can also determine the counit from the unit and vice versa directly, without going through the natural bijection:

Proposition. *Given the unit η of an adjunction $L \dashv R$, the counit ε is determined as the unique family of morphisms that makes this diagram commute:*



Proof (*in the locally small case*). Let θ be the natural bijection for this adjunction: we must have $\theta_{X,A}(f) = R(f) \circ \eta_X$. At the same time $\varepsilon_A = \theta^{-1}_{R(A),A}(\mathrm{id}_{R(A)})$. Therefore we have $R(\varepsilon_A) \circ \eta_{R(A)} = \mathrm{id}_{R(A)}$.

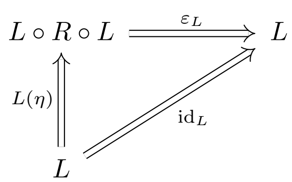


This is exactly the above diagram, applied to an arbitrary $A \in \mathrm{ob}(\mathcal{C})$. It uniquely determines ε_A because, since η is the unit of the adjunction, $(L(R(A)), \eta_{R(A)})$ is a universal morphism from $R(A)$ to R. □

There is also a counit-to-unit version, stated below.

Remark. These two commutative diagrams involving the unit and counit are called the *triangle identities*.

Proposition. *Given the counit ε of an adjunction $L \dashv R$, the unit η is determined as the unique family of morphisms that makes this diagram commute:*



Proof. Similar to the previous proof. □

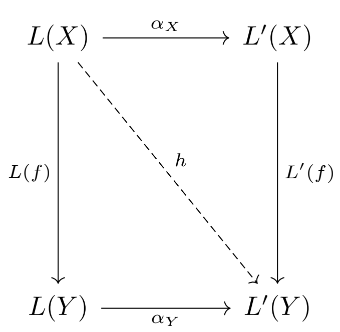
Theorem. *Conversely, if $\eta\colon \mathrm{Id}_{\mathcal{D}} \Rightarrow R \circ L$ and $\varepsilon\colon L \circ R \Rightarrow \mathrm{Id}_{\mathcal{C}}$ satisfy the triangle identities, then they are the unit and counit of an adjunction $L \dashv R$.*

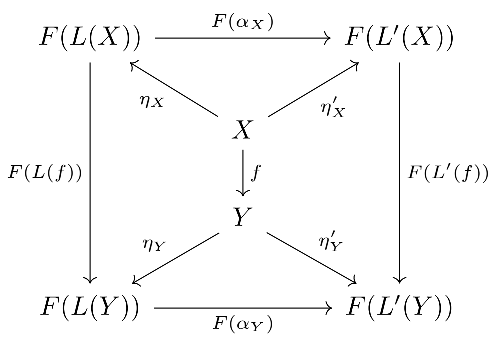
Proof idea (*locally small case*). Use the definitions of $\theta_{X,A}$ in terms of η and $\theta^{-1}_{X,A}$ in terms of ε, and show that they are natural and mutually inverse. □

Let us establish a few more properties of adjunctions.

Proposition (Adjoints are unique up to natural isomorphism)**.** *If $L, L'\colon \mathcal{D} \to \mathcal{C}$ are left adjoints to $F\colon \mathcal{C} \to \mathcal{D}$, then $L \cong L'$. The analogous property holds for right adjoints.*

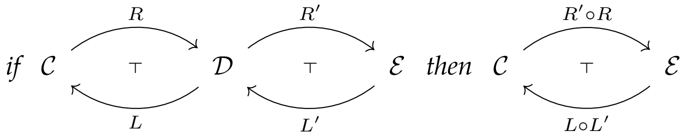
Proof. We use the uniqueness up to (unique) iso of universal morphisms to build a family $\alpha = (\alpha_X \in \mathrm{Iso}_{\mathcal{C}}(L(X), L'(X)))_{X \in \mathrm{ob}(\mathcal{C})}$. Then we need to show that α is natural. Consider the following:



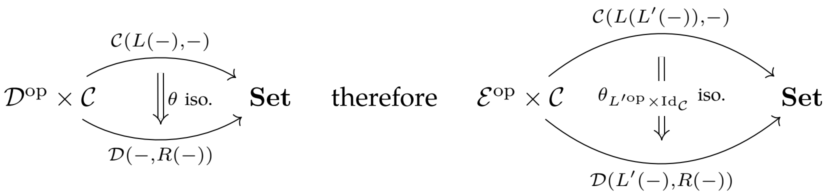


Exercise: meditate on this diagram to finish the proof (what property uniquely characterises h?). □

Proposition (Composition of adjunctions)**.** *If $L \dashv R$ and $L' \dashv R'$ with compatible types* (*cf. below*), *then $L \circ L' \dashv R' \circ R$.*



Proof. $\mathcal{C}(L(L'(X)), A) \cong \mathcal{D}(L'(X), R(A)) \cong \mathcal{E}(X, R'(R'(A)))$ naturally in A and X. More precisely:



realizes the first natural isomorphism, and similarly for the second one. □

Finally we show that adjoints interact nicely with (co)products.

Definition. A functor $F \colon \mathcal{C} \to \mathcal{D}$ *preserves products* if, whenever $(C, (\pi_i)_{i \in I})$ is a product of $(A_i)_{i \in I}$ in $\mathcal{C}$, then $(F(C), (F(\pi_i))_{i \in I})$ is a product of $(F(A_i))_{i \in I}$.

"F preserves coproducts" is defined analogously.

Theorem. *If a functor F is a right adjoint, then it preserves products.*

Dually, if a functor F is a left adjoint, then it preserves coproducts.

Remark. Beware: to *be* a right adjoint is to *have* a left adjoint.

There are many examples, let's talk about them before proving the theorem:

- The forgetful $U \colon \mathbf{Mon} \to \mathbf{Set}$ is right adjoint to $(-)^*$. Therefore, it must preserve products: $U(M \,\&\, M') \cong U(M) \times U(M')$. That is indeed the case: a product of two monoids in **Mon** is built from a $\times$ of their sets of elements (so this $\cong$ is a $=$ for the canonical cartesian structure on **Mon**).
- Since $(-)^*$ is a left adjoint, $(A + B)^*$ is a coproduct of A^* and B^* in **Mon**, as claimed in Lecture 5. For instance $\{a\}^* \oplus \{b\}^* \cong \{a, b\}^*$.
- A singleton monoid S is initial in **Mon**, but $U(S) \neq \varnothing$ is not initial in **Set**. Therefore U does not preserve 0-ary coproducts, so it is not a left adjoint / does not have a right adjoint!
- The forgetful $U_{\text{po}} \colon \mathbf{PreOrd} \to \mathbf{Set}$ is both a right adjoint (to the trivial preorder) and a left adjoint (to the discrete preorder). This is consistent with the fact that products of preorders can be built using products of sets, and coproducts of preorders can also be built using coproducts of sets.
- From the adjunction $(-) \times A \dashv \mathbf{Set}(A, -)$, we get
  - $\mathbf{Set}(A, B \times C) \cong \mathbf{Set}(A, B) \times \mathbf{Set}(A, C)$ (also a consequence of the characterisation of products using representable functors)
  - $(B + C) \times A \cong (B \times A) + (C \times A)$ – so even though this *distributivity* law of products over coproducts does not work in arbitrary categories, it has a categorical explanation in **Set**! (And in other *cartesian closed categories*, see Olivier Laurent's part of the course.)

These examples can also be stated for arbitrary families, e.g.

$$\left(\sum_{i \in I} X_i \right) \times A \cong \sum_{i \in I} X_i \times A_i$$