(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: LECTURE 7

3 OCTOBER 2024 — L. T. D. NGUYỄN

Last time: natural transformations, natural isomorphisms.

Remark. We write $F \cong G$ to say that they two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ are naturally isomorphic (indeed this is an isomorphism of objects in the category $[\mathcal{C}, \mathcal{D}]$).

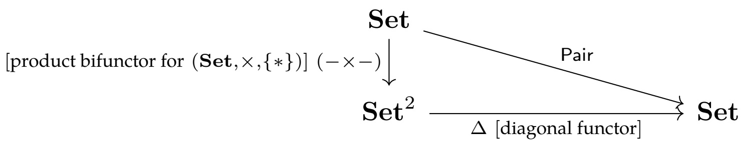
Representable functors

In short: representable functors are those naturally isomorphic to a hom-functor.

Definition. Let $\mathcal{C}$ be a locally small category and $F\colon \mathcal{C} \to \mathbf{Set}$ be a functor.

A *representation* of F is a pair (A, θ) where $A \in \mathrm{ob}(\mathcal{C})$ and $\theta\colon \mathcal{C}(A, -) \Rightarrow F$ is a natural isomorphism. If F admits a representation, it is said to be *representable*.

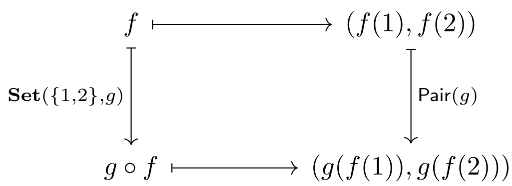
- Here is an example first alluded to in Lecture 3. Consider the endofunctor on $\mathbf{Set}$ of *pairs* $X \mapsto X^2$, also definable as



  It is represented by $(\{1,2\}, \theta)$ where

$$\theta_X\colon \mathbf{Set}(\{1,2\}, X) \to X^2$$
$$f \mapsto (f(1), f(2))$$

  Indeed, we can check that this defines a natural transformation:



  and since each θ_X is also bijective, θ is a natural isomorphism.
- The contravariant powerset functor $\mathcal{P}\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ (Lecture 4) is also representable — the following family of bijections is natural in X:

$$\mathbf{Set}^{\mathrm{op}}(\{\mathtt{yes}, \mathtt{no}\}, X) = \mathbf{Set}(X, \{\mathtt{yes}, \mathtt{no}\}) \to \mathcal{P}(X)$$
$$f \mapsto f^{-1}(\{\mathtt{yes}\})$$

  (the converse maps a subset of X to its indicator function).
- The forgetful functor $U\colon \mathbf{Mon} \to \mathbf{Set}$ is represented by $((\mathbb{N}, +, 0), \theta)$ where

$$\theta_{(M,\cdot,e)}\colon \mathbf{Mon}((\mathbb{N}, +, 0), (M, \cdot, e)) \to M$$
$$h \mapsto h(1)$$

  ($n \mapsto \overbrace{x \cdot \ldots \cdot x}^{n \text{ times}}$ is the unique homomorphism $\mathbb{N} \to M$ that maps 1 to x).

Representable functors are important because they can be *used to present universal properties*, as the following theorem shows.

Theorem. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally small categories.*

An object $A \in \mathrm{ob}(\mathcal{C})$ is part of a universal morphism from $X \in \mathrm{ob}(\mathcal{D})$ to $F\colon \mathcal{C} \to \mathcal{D}$ if and only if $\mathcal{C}(A,-) \cong \mathcal{D}(X, F(-)) = \mathcal{D}(X,-) \circ F$.

More precisely, there is an explicit bijection:

$$\{\textit{representations of } \mathcal{D}(X,F(-))\} \cong \{\textit{universal morphisms from } X \textit{ to } F\}$$
$$(A,\theta) \mapsto (A, \theta_A(\mathrm{id}_A))$$
$$\left(A, \begin{bmatrix} \theta_B\colon \mathcal{C}(A,B) \to \mathcal{D}(X,F(B)) \\ f \mapsto F(f)\circ\varphi \end{bmatrix}\right) \leftarrow\!\shortmid (A,\varphi)$$

Thus, for the forgetful functor $U\colon \mathbf{Mon} \to \mathbf{Set}$, we have a natural bijection in M

$$\mathbf{Mon}(X^*, M) \cong \mathbf{Set}(X, U(M)) = M^X$$
$$h \mapsto (x \mapsto h([x]))$$

and in particular $\mathbf{Mon}(\{a\}^*, M) \cong \mathbf{Set}(\{a\}, U(M)) \cong U(M)$ naturally in M. (Since $\{a\}^* \cong M$, this is consistent with our previous representation of U.)

As another example, let us take the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C}\times\mathcal{C}$. We have:

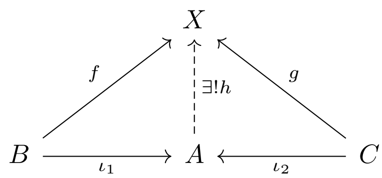
$$A \text{ is a coproduct of } B \text{ and } C \quad\iff\quad \mathcal{C}(A,-) \cong \underbrace{\mathcal{C}(B,-)\times\mathcal{C}(C,-)}_{=\,(\mathcal{C}\times\mathcal{C})((B,C),\Delta(-))}$$

We can get the natural bijection from the coprojections:

$$\theta_X\colon \mathcal{C}(A,X) \to \mathcal{C}(B,X)\times\mathcal{C}(C,X)$$
$$h \mapsto (h\circ\iota_1,\ h\circ\iota_2)$$

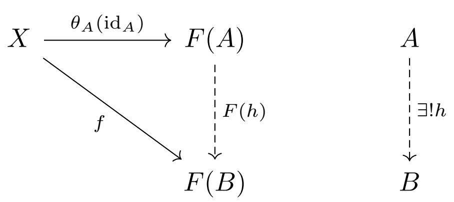
and in the converse direction $(\iota_1,\iota_2) = \theta_A(\mathrm{id}_A)$. The bijectivity of θ_X is precisely the universal property of the coproduct! And its inverse is the copairing map.

$$\exists! h : \theta_X(h) = (f,g) \qquad \iff$$



Now let us prove the theorem in general.

Representation $\mapsto$ universal morphism. Let $\theta\colon \mathcal{C}(A,-) \Rightarrow \mathcal{D}(X,F(-))$ be a natural isomorphism. We want to show the following universal property:



We have a single morphism in $\mathcal{C}$ involved in the above diagram, and we know that θ is a natural transformation between functors from $\mathcal{C}$, so it seems reasonable to consider the naturality square for this morphism (below, left). We then apply it to

id_A (below, right) — note that the reason it makes sense to work with "elements" in this commutative diagram is that it "lives" in the category **Set**!

$$\begin{array}{ccc} \mathcal{C}(A,A) & \xrightarrow{\theta_A} & \mathcal{D}(X,F(A)) \\ {\scriptstyle \mathcal{C}(A,h)}\downarrow & & \downarrow{\scriptstyle \mathcal{D}(X,F(h))} \\ \mathcal{C}(A,B) & \xrightarrow[\theta_B]{} & \mathcal{D}(X,F(B)) \end{array} \qquad \begin{array}{ccc} \mathrm{id}_A & \xmapsto{\theta_A} & \theta_A(\mathrm{id}_A) \\ & & \downarrow{\scriptstyle \mathcal{D}(X,F(h))} \\ {\scriptstyle \mathcal{C}(A,h)}\downarrow & & F(h)\circ\theta_A(\mathrm{id}_A) \\ & & \| \\ h\circ\mathrm{id}_A & \xmapsto[\theta_B]{} & \theta_B(h) \end{array}$$

Thanks to this equality, the universal property that we want can be rephrased as

$$\forall B \in \mathrm{ob}(\mathcal{D}),\ \forall f \in \mathcal{D}(X, F(B)),\ \exists! h \in \mathcal{C}(A,B) : \theta_B(h) = f$$

which is true because each θ_B is a bijection. □

Universal morphism $\mapsto$ representation. For (A,φ) a universal morphism from X to F,

$$\theta_B : \mathcal{C}(A,B) \to \mathcal{D}(X,F(B))$$
$$f \mapsto F(f)\circ\varphi$$

is bijective for all B because this is precisely the universal property satisfied by (A,φ), as illustrated by the case of coproducts above.

We still need to check naturality, i.e. that for every morphism $g \in \mathcal{C}(B,C)$, the corresponding naturality square commutes. Since it is in **Set**, it suffices to check that it commutes for every input element $f \in \mathcal{C}(A,B)$.

$$\begin{array}{ccc} \mathcal{C}(A,B) & \xrightarrow{\theta_B} & \mathcal{D}(X,F(B)) \\ {\scriptstyle \mathcal{C}(A,g)}\downarrow & & \downarrow{\scriptstyle \mathcal{D}(X,F(g))} \\ \mathcal{C}(A,C) & \xrightarrow[\theta_C]{} & \mathcal{D}(X,F(C)) \end{array} \qquad \begin{array}{ccc} f & \xmapsto{\theta_B} & F(f)\circ\varphi \\ & & \downarrow{\scriptstyle \mathcal{D}(X,F(g))} \\ {\scriptstyle \mathcal{C}(A,g)}\downarrow & & F(g)\circ F(f)\circ\varphi \\ & & \| \\ g\circ f & \xmapsto[\theta_C]{} & F(g\circ f)\circ\varphi \end{array}$$

Since F is a functor, $F(g\circ f) = F(g)\circ F(f)$ so the diagram commutes. □

These are inverse bijections. In one direction, $F(\mathrm{id}_A)\circ\varphi = \varphi$. In the other, it has been established previously that $F(h)\circ\theta_A(\mathrm{id}_A) = \theta_B(h)$. □

We also have a dual version with $\mathcal{D}(F(-),X) = \mathcal{D}(-,X)\circ F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

Theorem. *There is an explicit bijection:*

$$\{\textit{representations of } \mathcal{D}(F(-),X)\} \cong \{\textit{universal morphisms from } F \textit{ to } X\}$$
$$(A,\theta) \mapsto (A, \theta_A(\mathrm{id}_A))$$
$$\left(A, \begin{bmatrix} \theta_B : \mathcal{C}(B,A) \to \mathcal{D}(F(B),X) \\ f \mapsto \varphi\circ F(f) \end{bmatrix}\right) \mapsfrom (A,\varphi)$$

For example, (C,π_1,π_2) is a product of A and B if and only if the following natural transformation in X is bijective:

$$\mathcal{C}(X,C) \to \mathcal{C}(X,A)\times\mathcal{C}(X,B)$$
$$f \mapsto (\pi_1\circ f,\ \pi_2\circ f)$$

In a category representing a preordered set, this means that c is an infimum of a and b if and only if $\forall x,\ x \leqslant c \iff x \leqslant a$ and $x \leqslant b$.

Adjoint functors

Idea: global solutions to universal properties yield functors, generalizing

$$\text{choice of binary products} \quad \rightsquigarrow \quad \text{bifunctor } (- \,\&\, -)$$

Theorem. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and for each $X \in \mathrm{ob}(\mathcal{D})$, let $(L(X), \eta_X)$ be a chosen universal morphism from X to F.*

There is a unique extension of this operation $L\colon \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ to a functor $L\colon \mathcal{C} \to \mathcal{D}$ such that $(\eta_X \in \mathcal{C}(X, F(L(X)))_{X \in \mathrm{ob}(\mathcal{C})}$ is a natural transformation $\eta\colon \mathrm{Id}_{\mathcal{D}} \Rightarrow F \circ L$.

Proof. Let $f \in \mathcal{C}(X,Y)$. By the universal property of $(L(X), \eta_X)$, there is a unique way to define $L(f)$ that makes the naturality square commute — see the upper half of the diagram below:

$$\begin{array}{ccccc}
X & \xrightarrow{\eta_X} & F(L(X)) & \qquad & L(X) \\
\downarrow{\scriptstyle f} & & \vdots\, {\scriptstyle F(L(f))} & & \vdots\, {\scriptstyle L(f)} \\
Y & \xrightarrow[\eta_Y]{} & F(L(Y)) & & L(Y) \\
\downarrow{\scriptstyle g} & & \vdots\, {\scriptstyle F(L(g))} & & \vdots\, {\scriptstyle L(g)} \\
Z & \xrightarrow[\eta_Z]{} & F(L(Z)) & & L(Z)
\end{array}$$

In the lower half we have put the definition of $L(g)$; our goal in doing so is to show that L is a functor. Since the squares commute, the big rectangle commutes. By functoriality of F we have

$$\begin{array}{ccccc}
X & \xrightarrow{\eta_X} & F(L(X)) & \qquad & L(X) \\
\downarrow{\scriptstyle g \circ f} & & \vdots\, {\scriptstyle F(L(g) \circ L(f))} & & \vdots\, {\scriptstyle L(g) \circ L(f)} \\
Z & \xrightarrow[\eta_Z]{} & F(L(Z)) & & L(Z)
\end{array}$$

This commutative diagram is the one that defines $L(g \circ f)$; by uniqueness in the universal property of $(L(X), \eta_X)$, we get $L(g \circ f) = L(g) \circ L(f)$. One can also check that $L(\mathrm{id}_X) = \mathrm{id}_{L(X)}$. □

Definition. In this situation, the functor L is said to be (a) *left adjoint* to F and η is called the *unit of the adjunction*.

For example the free monoid functor $(-)^*\colon \mathbf{Set} \to \mathbf{Mon}$ is left adjoint to the forgetful functor $U\colon \mathbf{Mon} \to \mathbf{Set}$ because $x \in X \mapsto [x] \in U(X^*)$ is natural in X.

The point of view of representable functors is:

Proposition. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally small categories. The functor $L\colon \mathcal{D} \to \mathcal{C}$ is left adjoint to $F\colon \mathcal{C} \to \mathcal{D}$ if and only if $\mathcal{C}(L(A), X) \cong \mathcal{D}(A, F(X))$ naturally in A and X:*

$$\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \;\begin{array}{c} \xrightarrow{\mathcal{C}(L(-),-) = \mathcal{C}(-,-) \circ (L^{\mathrm{op}} \times \mathrm{Id}_{\mathcal{C}})} \\ \Downarrow\, \theta \text{ iso} \\ \xrightarrow[\mathcal{D}(-,F(-))]{} \end{array}\; \mathbf{Set}$$

Proof idea. Via the characterisation of universal morphisms by representable functors, L is left adjoint to F if and only if, for each $X \in \mathrm{ob}(\mathcal{C})$, there is a natural iso

$$(\theta_{X,A})_{A \in \mathrm{ob}(\mathcal{C})}\colon \mathcal{C}(L(X), -) \Rightarrow \mathcal{D}(X, R(-))$$

such that $\eta = (\theta_{X,L(X)}(\mathrm{id}_{L(X)}))$ is a natural transformation $\mathrm{Id}_{\mathcal{D}} \Rightarrow F \circ L$. Therefore the equivalence that we want to show is: given these $\theta_{X,A}$ that are natural in A,

$$\theta \text{ is also natural in } X \qquad \iff \qquad \eta \text{ is natural}$$

In the direction ($\implies$), we have: for any $f \in \mathcal{D}(X, Y)$,

$$\begin{aligned}
\eta_Y \circ f &= \theta_{Y,L(Y)}(\mathrm{id}_{L(Y)}) \circ f \\
&= \mathcal{D}(f, L(F(Y)))(\theta_{Y,L(Y)}(\mathrm{id}_{L(Y)})) \\
&= \theta_{X,L(Y)}(\mathcal{C}(L(f), F(Y))(\mathrm{id}_{L(Y)})) \quad \text{by naturality} \\
&= \theta_{X,L(Y)}(\mathrm{id}_{L(Y)} \circ L(f)) \\
&= F(L(f)) \circ \eta_X
\end{aligned}$$

where the last step uses the formula for $\theta_{X,A}$ in terms of F and η_X that we have seen earlier (this formula is a consequence of naturality in A).

The converse is left as an exercise. □

Of course, for a functor $F' : \mathcal{C}' \to \mathcal{D}'$, by applying the previous theorems to G^{op} we get the dual statements:

- a choice $(R(A), \varepsilon_A)$ of universal morphisms from F' to each $A \in \mathrm{ob}(\mathcal{D}')$ extends uniquely to a functor $R : \mathcal{D}' \to \mathcal{C}'$ making $\varepsilon : F' \circ R \Rightarrow \mathrm{Id}_{\mathcal{D}'}$ natural
  - R is then called a *right adjoint* to F'
  - and ε the *counit* of the adjunction
- a functor R is right adjoint to F if and only if $\mathcal{C}'(X, R(A)) \cong \mathcal{D}'(F'(X), A)$ naturally in X and A.

The key observation is to compare the characterisations of left and right adjoints based on natural bijections. By taking $L = F'$ and $R = F$, we see that[1]:

Corollary. *L is left adjoint to R if and only if R is right adjoint to L.*

This is a non-trivial coincidence: in the definition of "L is left adjoint to R" via universal morphisms, the two functors play highly asymmetric roles (R specifies a "problem" and L gives a "solution"), so there is no obvious reason that dualizing this definition via $(-)^{\mathrm{op}}$ would amount to the same thing as swapping the roles of L and R!

Definition. In this situation, we say that "L and R are adjoint" and write $L \dashv R$.

[1] For locally small categories, but we admit this is true in general.