(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: LECTURE 6

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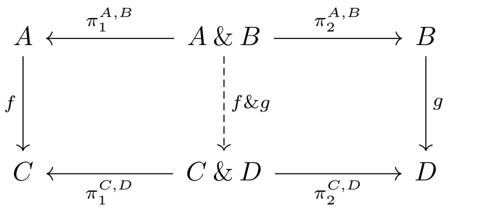
Last time: a cartesian category $(\mathcal{C}, \&, \top)$ is a category $\mathcal{C}$ equipped with a choice of binary products and of a terminal object. We saw that '$\&$' is associative, commutative and unital up to isomorphism, and can be used to construct n-ary products.

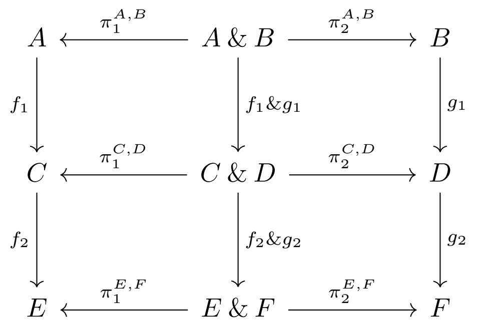
Now, let us show that it is *functorial*.

Definition. Let $(\mathcal{C}, \&, \top)$ be a cartesian category, $A, B, C, D \in \mathrm{ob}(\mathcal{C})$, $f \in \mathcal{C}(A, C)$ and $g \in \mathcal{C}(B, D)$. We define $f \,\&\, g \in \mathcal{C}(A \,\&\, B,\ C \,\&\, D)$ as $\langle f \circ \pi_1^{A,B},\ g \circ \pi_2^{A,B} \rangle$.

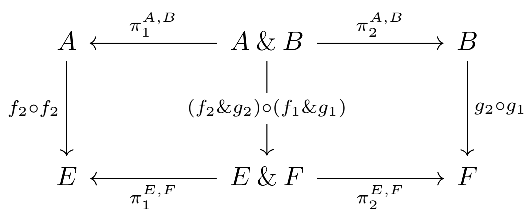


Proposition. *This defines a bifunctor* $(- \,\&\, -) \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Proof. $\mathrm{id}_A \,\&\, \mathrm{id}_B = \mathrm{id}_{A\&B}$ is immediate from uniqueness in the universal property of the product $A \,\&\, B$. For composition, we look at this diagram, which commutes because its upper and lower halves commute:



Forgetting the middle line, we get



By definition, $(f_2 \circ f_1) \,\&\, (g_2 \circ g_1)$ is the only morphism from $A \,\&\, B$ to $C \,\&\, D$ that makes the diagram commute, so

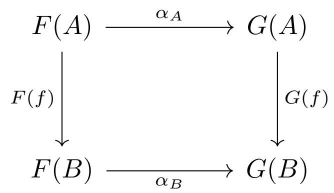
$$(f_2 \circ f_1) \,\&\, (g_2 \circ g_1) = (f_2 \,\&\, g_2) \circ (f_1 \,\&\, g_1)$$

which means that $(- \,\&\, -)$ preserves composition. $\square$

Natural transformations

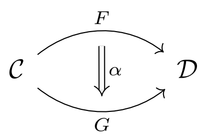
Motivation: in a cartesian category $(\mathcal{C}, \&, \top)$, we have families of morphisms $\pi_1^{A,B} \in \mathcal{C}(A \& B,\ A)$, $\sigma_{A,B} \in \mathcal{C}(A \& B,\ B \& A)$, ... which are "generic" / "uniformly defined" over all objects A, B. Thanks to this, they satisfy some compatibility condition with the bifunctor $(- \& -)$.

Definition. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* from F to G is a family $\alpha = (\alpha_A \in \mathcal{D}(F(A), G(A)))_{A \in \mathrm{ob}(\mathcal{C})}$ such that for any $A, B \in \mathrm{ob}(\mathcal{C})$ and $f \in \mathcal{C}(A, B)$, the following diagram ("naturality square") commutes:



i.e. $\alpha_B \circ F(f) = G(f) \circ \alpha_A$

We write $\alpha\colon F \Rightarrow G$, or in a diagram:



For example $(\pi_1^{A,B})_{A,B \in \mathrm{ob}(\mathcal{C})}$ defines a natural transformation $\pi_1\colon (-\&-) \Rightarrow \Pi_1$

$$\text{where } \Pi_1\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
$$(A, B) \mapsto A$$
$$(f, g) \mapsto f$$

Proof. The left half of the diagram defining $f \& g$ is a naturality square for π_1! □

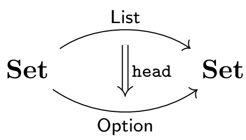
To avoid explicitly writing the bifunctors, we also say that

"$\pi_1^{A,B} \in \mathcal{C}(A \& B,\ A)$ is *natural in A and B*".

Other examples include *polymorphic* functions between generic data structures that are represented as endofunctors on **Set**. (For more on the connection between polymorphism and naturality, cf. [Wad89, HRR13].) For instance:

$$\mathtt{head}_A\colon \mathsf{List}(A) \to \mathsf{Option}(A)$$
$$[] \mapsto \mathsf{None}$$
$$[a_1, \dots, a_n] \mapsto \mathsf{Some}(a_1) \text{ for } n \geqslant 1$$

defines a natural transformation



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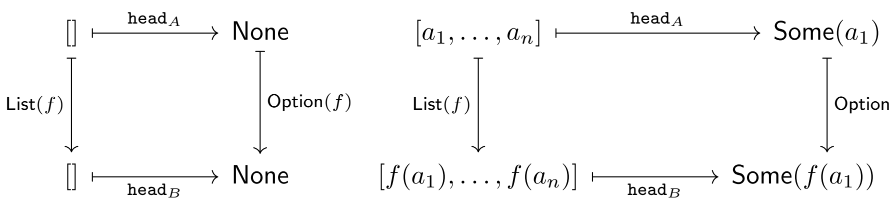
Proof. For $f\colon A \to B$, we can check the naturality square by case analysis:

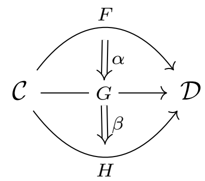


□

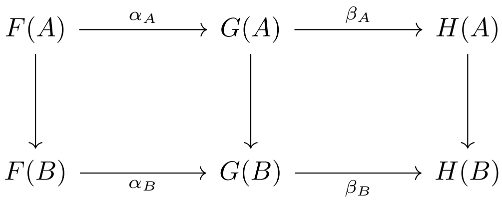
We can build new natural transformations out of new ones. A simple way is:

Definition. Let $F, G, H \colon \mathcal{C} \to \mathcal{D}$ be functors and $\alpha \colon F \Rightarrow G$, $\beta \colon G \Rightarrow H$ be natural transformations. We define $\beta \circ \alpha = (\beta_A \circ \alpha_A)_{A \in \mathrm{ob}(\mathcal{C})}$.

This is called *vertical composition* because of the shape of this diagram:



To see that $\beta \circ \alpha$ is a natural transformation $F \Rightarrow H$, observe that in the commutative diagram below, the small squares commute by naturality of α and β, and the outer rectangle is the naturality condition for $\beta \circ \alpha$:

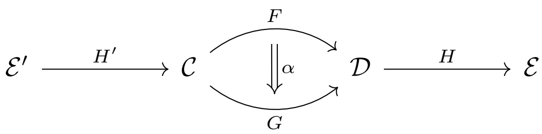


In equations: $\beta_B \circ \alpha_B \circ F(f) = \beta_B \circ G(f) \circ \alpha_A = H(f) \circ \beta_A \circ \alpha_A$.

We also have another kind of composition, but to define it, we first need to define how natural transformations can interact with functors.

Definition ("whiskering"[1]). Let $F, G \colon \mathcal{C} \to \mathcal{D}$ be functors and $\alpha \colon F \Rightarrow G$.

- For $H \colon \mathcal{D} \to \mathcal{E}$, we define $H(\alpha) = (H(\alpha_A))_{A \in \mathrm{ob}(\mathcal{C})} \colon H \circ F \Rightarrow H \circ G$.
- For $H' \colon \mathcal{E}' \to \mathcal{C}$, we define $\alpha_{H'} = (\alpha_{H'(X)})_{X \in \mathrm{ob}(\mathcal{E}')} \colon F \circ H' \Rightarrow G \circ H'$.



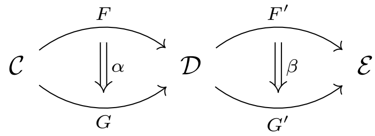
These are natural transformations because:

- The naturality squares for $H(\alpha)$ are images by H — which, being a functor, preserves commutative diagrams — of those for α.
- The naturality squares for α_H are special cases of those for α.

Proposition. $H(\alpha_{H'}) = (H(\alpha))_{H'}$.

Proof. For all $X \in \mathrm{ob}(\mathcal{E}')$, the X-components of the two natural transformations are both equal to $H(\alpha_{H'(X)})$ by definition. $\square$

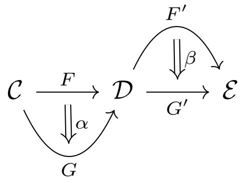
Proposition. *Let α and β be natural transformations with the following types:*

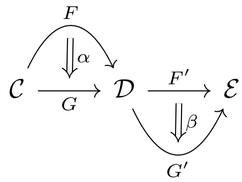


Then $G'(\alpha) \circ \beta_F = \beta_G \circ F'(\alpha)$.

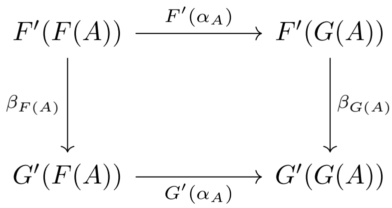
[1] This is the actual name: `https://ncatlab.org/nlab/show/whiskering` — comparing the shape of the diagram to a cat's whiskers is left to the reader's imagination.

This expresses the fact that the two obvious ways to put α and β together, illustrated respectively by the two diagrams below, coincide.





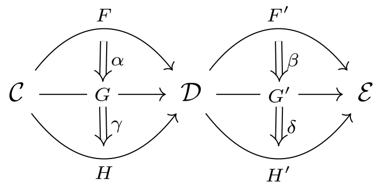
Proof. This equality corresponds to the diagram



which is a naturality square for β, and therefore commutes. □

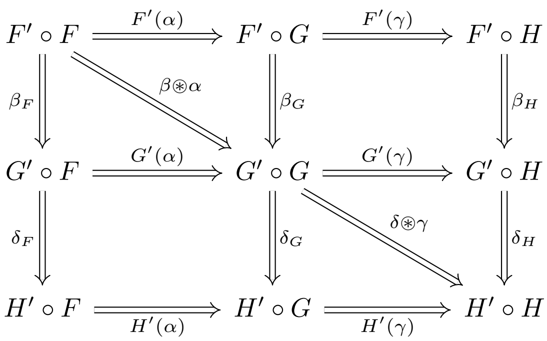
Definition. We set $\beta \circledast \alpha = G'(\alpha) \circ \beta_F = \beta_G \circ F'(\alpha)$ and call it the *horizontal composition* of α and β.

Theorem (Interchange law)**.** *Let $\alpha, \beta, \gamma, \delta$ be natural transformations with the types:*



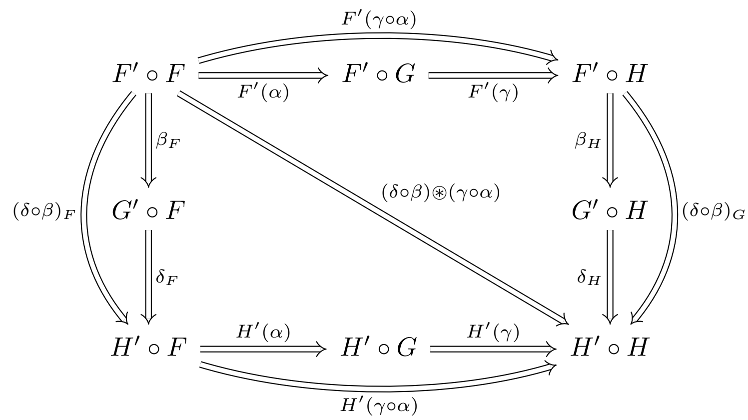
We have $(\beta \circledast \alpha) \circ (\delta \circledast \gamma) = (\delta \circ \beta) \circledast (\gamma \circ \alpha)$. In other words, the diagram describes only one natural transformation of type $F' \circ F \Rightarrow G' \circ G$.

Proof. Consider the following diagram:



The 4 small squares commute because they are naturality squares for either β or δ, as in the proof of the previous proposition. The small triangles commute by definition of $\circledast$. Therefore the whole diagram commutes.

Furthermore, by definition of $(\delta \circ \beta) \circledast (\gamma \circ \alpha)$ and functoriality of F and H, the following diagram also commutes:



Since the big squares of the two diagrams coincide, the natural transformations of type $F' \circ F \Rightarrow H' \circ H$ that they define are equal, giving us the equality we want. □

In the above proof, we have used commutative diagrams whose arrows denote natural transformations, where sequencing of arrows is vertical composition. Do these diagrams live in a category? Observe that all functors that appear in those diagrams have the same type $\mathcal{C} \to \mathcal{E}$.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. The *category of functors* $\overbrace{[\mathcal{C}, \mathcal{D}]}^{\text{notation already used for the collection of functors } \mathcal{C}\to\mathcal{D}}$ has:

- **as objects:** all functors $\mathcal{C} \to \mathcal{D}$
- **as morphisms from F to G:** all natural transformations $F \Rightarrow G$
- **as composition:** vertical composition

Note that vertical composition is associative because it is defined componentwise, and composition in $\mathcal{C}$ is associative. There are also identities: $\mathrm{id}_F = (\mathrm{id}_{F(A)})_{A \in \mathrm{ob}(\mathcal{C})}$.

Since $[\mathcal{C}, \mathcal{D}]$ is a category, it makes sense to talk about its isomorphisms:

Definition. A *natural isomorphism* from F to G is an element of $\mathrm{Iso}_{[\mathcal{C},\mathcal{D}]}(F, G)$.

Before looking an example, let us establish a property that makes it easier to check that a family of morphisms is a natural isomorphism.

Proposition. *Let $F, G \colon \mathcal{C} \to \mathcal{D}$. If a natural transformation $\alpha \colon F \Rightarrow G$ is such that α_A is an isomorphism for every $A \in \mathrm{ob}(\mathcal{C})$, then α is a natural isomorphism.*

Proof. For $f \in \mathcal{C}(A, B)$, starting from the naturality of α, we have:

$$\begin{aligned} \alpha_B \circ F(f) &= G(f) \circ \alpha_A \\ \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} &= \alpha_B^{-1} \circ G(f) \circ \alpha_A \circ \alpha_A^{-1} \\ F(f) \circ \alpha_A^{-1} &= \alpha_B^{-1} \circ G(f) \end{aligned}$$

and the last equation says that $\alpha^{-1} = (\alpha_A^{-1})_{A \in \mathcal{C}}$ is a natural transformation. We can check that α^{-1} is an inverse to α in $[\mathcal{C}, \mathcal{D}]$ (by computing $\alpha \circ \alpha^{-1}$ and $\alpha^{-1} \circ \alpha$ componentwise). Therefore, $\alpha \in \mathrm{Iso}_{[\mathcal{C},\mathcal{D}]}(F, G)$. □

As an example, the isomorphisms $\sigma_{A,B} \in \mathcal{C}(A \,\&\, B,\ B \,\&\, A)$ that we defined in the previous lecture are also natural in A and B (exercise). Therefore, they form a natural isomorphism from $(- \,\&\, -)$ to $(- \,\&\, -) \circ \mathrm{Flip}$ where $\mathrm{Flip} \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ maps (A, B) to (B, A) and (f, g) to (g, f).

References

[HRR13] Claudio Hermida, Uday S. Reddy, and Edmund P. Robinson. Logical Relations and Parametricity – A Reynolds Programme for Category Theory and Programming Languages. In John Power and Cai Wingfield, editors, *Proceedings of the Workshop on Algebra, Coalgebra and Topology, WACT 2013, Bath, UK, March 1, 2013*, volume 303 of *Electronic Notes in Theoretical Computer Science*, pages 149–180. Elsevier, 2013. `doi:10.1016/j.entcs.2014.02.008`.

[Wad89] Philip Wadler. Theorems for free! In Joseph E. Stoy, editor, *Proceedings of the fourth international conference on Functional programming languages and computer architecture, FPCA 1989, London, UK, September 11-13, 1989*, pages 347–359. ACM, 1989. `doi:10.1145/99370.99404`.