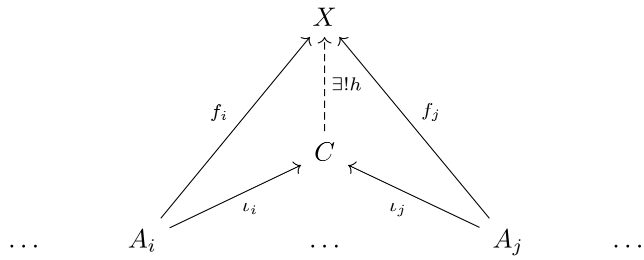
(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: LECTURE 5

26 SEPTEMBER 2024 — L. T. D. NGUYỄN

Last time: C is a *coproduct* of $(A_i)_{i \in I}$ in $\mathcal{C}$ with the *coprojections* $\iota_i \in \mathcal{C}(A_i, C)$ when:

$$\forall X \in \mathrm{ob}(\mathcal{C}),\ \forall (f_i \in \mathcal{C}(A_i, X))_{i \in I},\ \exists! h \in \mathcal{C}(C, X) : \forall i \in I,\ f_i = h \circ \iota_i$$



In Set: Last time we mentioned that the disjoint sum of two sets is a binary coproduct. More generally, a coproduct of a family $(A_i)_{i \in I}$ of sets is given by the dependent sum $\sum_{i \in I} A_i$ (as defined in the previous lecture) with the coprojections $\iota_i \colon a \mapsto (i, a)$.

In Rel: We can also build a coproduct using a dependent sum, with the relational coprojections $\iota_i^{\mathbf{Rel}} = \{(a, (i, a)) \mid a \in A_i\}$.

In PreOrd and Ord: A product of $(X_i, \leqslant_i)$ is given by $\left(\sum_{i \in I} X_i,\ \leqslant \right)$ where ι_i is the same as for **Set** and $(i, x) \leqslant (j, y) \iff i = j$ and $x \leqslant_i y$.

In $\mathcal{C}_{(X,\leqslant)}$: Dually to "product = infimum" for categories representing preorders, we have "coproduct = supremum".

In Mon: A family of monoids always admits a coproduct, but it has a slightly complicated construction using alternating lists (called the "free product" of monoids). However, the special case of *free monoids* is nice:

Proposition. *$(X + Y)^*$ is a coproduct of X^* and Y^* with the coprojections ι_1^* and ι_2^* (i.e. the images, by the free monoid functor, of the coprojections in* **Set***).*

This will be proved later — it is an instance of the fact that "left adjoints preserve coproducts".

Remark. Although we have only seen positive examples, (co)products do not always exist! As an example, consider the category $\mathcal{C}_{(\{a,b\},=)}$ induced by 2 elements with the equality preorder. The two elements (objects) a and b do not have any common lower bound, and therefore they have no infimum (product); nor do they have a supremum (coproduct).

Remark. Unary (co)products always exist: (A, id_A) is both the product and the coproduct of the family (A).

As for 0-ary coproducts, they are of course dual to 0-ary products, i.e. terminal objects.

Definition. An object C is *initial* in $\mathcal{C}$ if $\mathrm{card}(\mathcal{C}(C, X)) = 1$ for every $X \in \mathrm{ob}(\mathcal{C})$.

Proposition. *Initial object in $\mathcal{C}$ = terminal object in $\mathcal{C}^{\mathrm{op}}$ (and vice versa).*

In Set/Rel/PreOrd/Ord: The empty set is initial: for any set X there is a unique function $\varnothing \to X$, namely the "empty function". (In other words $X^{\varnothing}$ is a singleton, just as $x^0 = 1$ for a number x.)

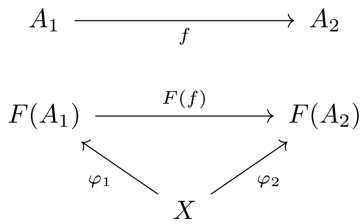
In Mon: Any singleton $\{x\}$, with the only possible monoid structure (that is, $x \cdot x = x$ and $e = x$) is initial! That is because for any monoid M, since monoid homomorphisms preserve the unit, the unique morphism in $\mathbf{Mon}(\{x\}, M)$ is $h\colon x \mapsto e_M$.

In $\mathcal{C}_{(X,\leqslant)}$: initial object = minimum element.

Finally, here is a more "meta" example of initial object.

Proposition. *Let $X \in \mathrm{ob}(\mathcal{D})$ and $F\colon \mathcal{C} \to \mathcal{D}$ a functor. Let $X \downarrow F$ be defined as:*

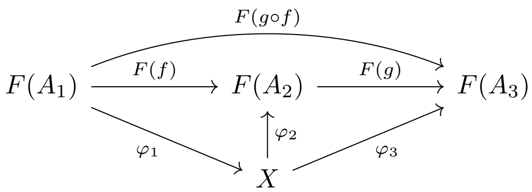
- $\mathrm{ob}(X \downarrow F) = \{(A, \varphi) \mid A \in \mathrm{ob}(\mathcal{C}),\ \varphi \in \mathcal{C}(X, F(A))\}$
- $(X \downarrow F)\big((A_1, \varphi_1), (A_2, \varphi_2)\big) = \{f \in \mathcal{C}(A_1, A_2) \mid F(f) \circ \varphi_1 = \varphi_2\}$



- *composition and identities are inherited from the category $\mathcal{C}$*

This is a category, whose initial objects are exactly the universal morphisms from X to F.

Proof. To check that it is a category we have to establish that composition is well-defined. The idea is that if $f \in \mathcal{C}(A_1, A_2)$ and $g \in \mathcal{C}(A_2, A_3)$ make the suitable triangles commute, then so does $g \circ f$, i.e. $F(g \circ f) \circ \varphi_1 = \varphi_3$. This works because the functor F preserves composition; the argument is summed up by the following commutative diagram:



The claim about initial objects is immediate by unfolding the definitions (you are highly encouraged to do this on a piece of paper!). □

Dually, universal morphisms from F to X can be described as terminal objects in a certain category $F \downarrow X$.

Uniqueness up to unique isomorphism

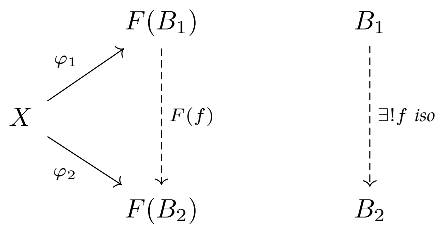
In the first lecture we claimed that universal properties characterise objects up to isomorphism, with the example of the free monoid. We shall now see general versions of this claim. In fact, an isomorphism is not just a property $A \cong B$, but a piece of *data* $f \in \mathrm{Iso}_{\mathcal{C}}(A, B)$, and we'll see that the data itself can be forced to be unique under some conditions!

Proposition. *Between two initial objects I_1 and I_2 of a category $\mathcal{C}$, there exists a unique isomorphism.*

Proof. Since I_1 is initial there is a morphism $f \in \mathcal{C}(I_1, I_2)$. Since I_2 is initial there is a morphism $g \in \mathcal{C}(I_2, I_1)$. They form a pair of inverse isomorphisms: indeed, $g \circ f$ and id_{I_1} are both in $\mathcal{C}(I_1, I_1)$ and I_1 is initial, so, by uniqueness, $g \circ f = \mathrm{id}_{I_1}$; similarly, $f \circ g = \mathrm{id}_{I_2}$. Finally, f is unique again because I_1 is initial. □

For other kinds of universal properties, the isomorphism is only unique when subject to an extra condition.

Theorem. *For any two universal morphisms (B_1, φ_1) and (B_2, φ_2) from $X \in \mathrm{ob}(\mathcal{D})$ to $F\colon \mathcal{C} \to \mathcal{D}$, there exists a unique $f \in \mathrm{Iso}_{\mathcal{C}}(B_1, B_2)$ such that $\varphi_2 = F(f) \circ \varphi_1$.*



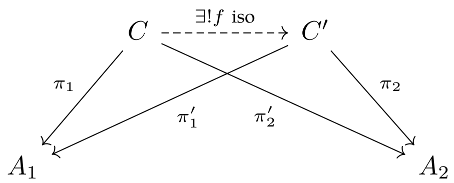
Proof. Apply the previous proposition to $X \downarrow F$. □

Alternative direct proof. There is a unique morphism f making this diagram commute because (B_1, φ_1) is a universal morphism. To show that $f \in \mathrm{Iso}_{\mathcal{C}}(B_1, B_2)$, one can adapt the proof for the free monoid (Lecture 1). □

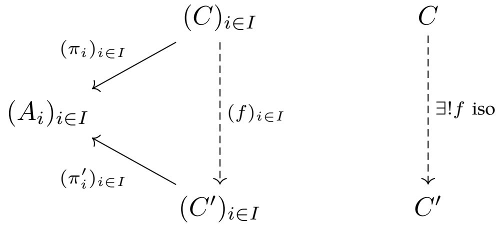
By duality, terminal objects and universal morphisms from F to X are also unique up to unique isomorphism.

Corollary. *If $(C, (\pi_i))$ and $(C', (\pi'_i))$ are two products of (A_i) then there exists a unique isomorphism $f \in \mathcal{C}(C, C')$ such that $\forall i \in I,\ \pi'_i \circ f = \pi_i$.*

Let us illustrate this with a commutative diagram for the binary case:



Proof. Apply the previous theorem to the diagonal functor Δ:



and observe that, since composition is performed componentwise in $\mathcal{C}^I$, what the diagram says is exactly that $\forall i \in I,\ \pi'_i \circ f = \pi_i$. □

Dually, coproducts are also unique up to unique isomorphism.

Cartesian categories

Suppose we have a category that "has all binary products": any two objects $A, B \in \mathrm{ob}(\mathcal{C})$ admit a product. We would like to refer to "the" product but there is still some ambiguity even though it is only up to unique isomorphism. Hence:

Definition. A *cartesian category* $(\mathcal{C}, \&, \top)$ is a category $\mathcal{C}$ endowed with:

- a *chosen* product $(A \& B,\ \pi_1^{A,B},\ \pi_2^{A,B})$ for any two objects $A, B \in \mathrm{ob}(\mathcal{C})$;
- a *chosen* terminal object $\top$.

Examples include $(\mathbf{Set}, \times, \{*\})$, $(\mathbf{Mon}, \times, \{*\})$, $(\mathbf{Rel}, +, \varnothing)$ and so on: in all our previous examples of existence of products in various categories, we exhibited a "standard" construction for the product.

As always, this definition can be dualised:

Definition. A *cocartesian category* $(\mathcal{C}, \oplus, 0)$ is a category endowed with chosen coproducts $(A \oplus B,\ \iota_1^{A,B},\ \iota_2^{A,B})$ and a chosen initial object 0.

Proposition. *A cocartesian structure* $(\oplus, 0)$ *on* $\mathcal{C}$ = *a cartesian structure* $(\&, \top)$ *on* $\mathcal{C}^{\mathrm{op}}$.

This means that any property concerning cartesian categories has a dual cocartesian counterpart: just reverse the arrows! We only focus here on the cartesian case.

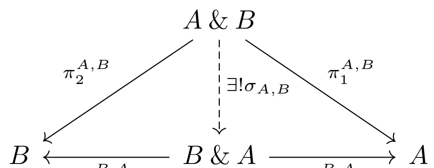
Remark. In the literature, the product operator in an arbitrary cartesian category is often denoted by $\times$. We reserve this notation for the usual product of sets (or sets with structure), and write $\&$ for an arbitrary product.

The notations $\&, \top, \oplus, 0$ are taken from linear logic, in order to be consistent with Olivier Laurent's part of the course. In this context there are debatable but sensible reasons to not write e.g. $\bot$ for the chosen initial object. (J.-Y. Girard, who introduced linear logic, has some comments in very bad taste about this debate on notations in [Gir11, Section 9.2.3].)

Multiplying numbers is an associative and commutative operation, with a unit element. For categorical products, we can hope for the same properties, but only up to isomorphism.

Proposition. *In a cartesian category,* $A \& B \cong B \& A$*. In other words, "*$\&$ *is commutative up to isomorphism".*

First "uninspired" proof to illustrate manipulating products. Let $\sigma_{A,B} = \langle \pi_2^{A,B}, \pi_1^{A,B} \rangle$ where $\langle -, - \rangle$ denotes the pairing for $A \& B$.



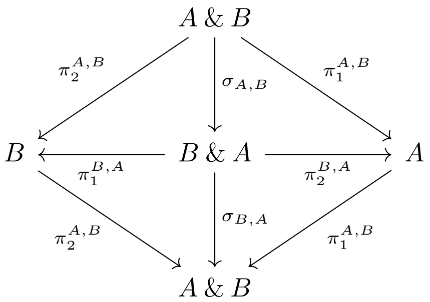
Let us show that $\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A\&B}$. We first determine $\pi_i^{A,B} \circ \sigma_{B,A} \circ \sigma_{A,B}$ for $i \in \{1, 2\}$. For $i = 1$, we expand the definitions of $\sigma_{A,B}$ and $\sigma_{B,A}$, and calculate:

$$\pi_1^{A,B} \circ \langle \pi_2^{B,A}, \pi_1^{B,A} \rangle \circ \langle \pi_2^{A,B}, \pi_1^{A,B} \rangle = \pi_2^{B,A} \circ \langle \pi_2^{A,B}, \pi_1^{A,B} \rangle = \pi_1^{A,B}$$

using the computation rule for a product (which holds by definition of $\langle -, - \rangle$)

$$\pi_i \circ \langle f_1, f_2 \rangle = f_i$$

Similarly we also have $\pi_2^{A,B} \circ \sigma_{B,A} \circ \sigma_{A,B} = \pi_2^{A,B}$. In fact, these two computations can be read on the following diagram — it commutes because its upper half and lower half both commute by definition of $\sigma_{A,B}$ and $\sigma_{B,A}$ respectively:



The left and right halves of the diagram tell us the equalities that we just stated concerning $\pi_i^{A,B} \circ \sigma_{B,A} \circ \sigma_{A,B}$, for $i = 2$ and $i = 1$ respectively. Furthermore:

$$\pi_2^{A,B} \circ \mathrm{id}_{A\&B} = \pi_2^{A,B} \qquad \pi_1^{A,B} \circ \mathrm{id}_{A\&B} = \pi_1^{A,B}$$

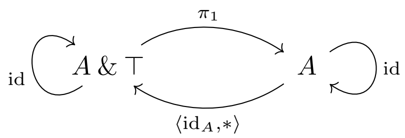
By uniqueness in the universal property of $A \& B$, we have $\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A\&B}$.

Likewise, $\sigma_{A,B} \circ \sigma_{B,A} = \mathrm{id}_{B\&A}$. Hence $\sigma_{A,B} \in \mathrm{Iso}_{\mathcal{C}}(A \& B,\ B \& A)$. □

Second "clever" proof. First, observe that (C, π_1, π_2) is a product of A and B if and only if (C, π_2, π_1) is a product of B and A (immediate by comparing the universal properties defining both). Thus, in particular, $(A \& B,\ \pi_2^{A,B},\ \pi_1^{A,B})$ is a product of B and A. Since products are unique up to (unique) iso, $A \& B \cong B \& A$. □

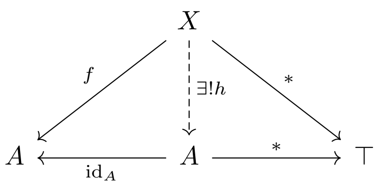
Proposition (Unitality up to iso)**.** *In a cartesian category,* $A \& \top \cong A \cong \top \& A$*.*

Proof idea 1. Show that this diagram commutes



where $*$ is the unique morphism in $\mathcal{C}(A, \top)$. □

Proof idea 2. We show that A is a product of A and $\top$, and therefore isomorphic to the chosen product $A \& \top$:



The left half of the diagram commutes if and only if $h = f$. The right half *always* commutes: two morphisms from a common source to a common terminal object are automatically equal. So there is indeed a unique solution for h. □

Proposition (Associativity up to iso)**.** $(A \& B) \& C \cong A \& (B \& C)$.

(We skip the proof.) In fact, we can say more: $(A \& B) \& C$ and $A \& (B \& C)$ are both products of the family $\{A, B, C\}$ — cf. Exercise 1 in Homework 3.

So cartesian categories admit all ternary products — actually, they have *all finite products* (again, we omit the proof):

Proposition. *For all* $n \in \mathbb{N}$, $A_1 \& \cdots \& A_n$ *is a product of* $(A_i)_{1 \leqslant i \leqslant n}$.

(*For* $n = 0$*: take the chosen terminal object* $\top$*.*)

References

[Gir11] Jean-Yves Girard. *The Blind Spot: Lectures on logic*. European Mathematical Society, September 2011. doi:10.4171/088.