(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: LECTURE 4

23 SEPTEMBER 2024 — L. T. D. NGUYỄN

Last time: duality and contravariant functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, which satisfy

$$F(g \circ_{\mathcal{C}} f) = F(f \circ_{\mathcal{C}^{\mathrm{op}}} g) = F(f) \circ_{\mathcal{D}} F(g)$$

e.g. the contravariant hom-functor $\mathcal{C}(-,X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ (for $\mathcal{C}$ locally small).

$$\mathcal{C}(g \circ_{\mathcal{C}} f,\, X)(h) = h \circ g \circ f = \mathcal{C}(g,X)(h) \circ f = \mathcal{C}(f,X)(\mathcal{C}(g,X)(h))$$

Other *examples*:

Contravariant powerset functor: $\begin{cases} \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set} \\ A \in \mathrm{ob}(\mathbf{Set}) \mapsto \mathcal{P}(A) \\ f \in \mathbf{Set}(A,B) \mapsto \begin{pmatrix} \mathcal{P}(B) \to \mathcal{P}(A) \\ X \mapsto f^{-1}(X) \end{pmatrix} \end{cases}$

Indeed, we have

$$\forall f\colon A \to B,\ \forall g\colon B \to C, \forall X \subseteq C,\ (g \circ f)^{-1}(X) = f^{-1}(g^{-1}(X))$$

On monoids: For a monoid M, we have $(\mathcal{C}_M)^{\mathrm{op}} = \mathcal{C}_{M^{\mathrm{op}}}$ where M^{op} is a monoid with the same set of elements and the same unit element as M, and binary operation $x \cdot_{M^{\mathrm{op}}} y = y \cdot_M x$. Therefore, a functor $(\mathcal{C}_M)^{\mathrm{op}} \to \mathcal{C}_N$ is "the same thing" as an *anti-homomorphism*, i.e. a map $h\colon M \to N$ such that $h(x \cdot y) = h(y) \cdot h(x)$ — for example, `reverse`: $X^* \to X^*$.

On preorders: $(\mathcal{C}_{(X,\leqslant)})^{\mathrm{op}} = \mathcal{C}_{(X,\geqslant)}$ so contravariant functors correspond to order-reversing maps (in French: "fonctions décroissantes").

We also introduced *bifunctors* $\overbrace{\mathcal{C} \times \mathcal{D}}^{\text{product of two categories}} \to \mathcal{E}$ last time. $\mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ examples:

Pairs: $\begin{cases} (A,B) \mapsto A \times B \\ (f,g) \mapsto \big((a,b) \mapsto (f(a), g(b))\big) \end{cases}$

Disjoint sum: $\begin{cases} (A,B) \mapsto A + B = (\{1\} \times A) \cup (\{2\} \times B) \\ (f,g) \mapsto \begin{pmatrix} (1,a) \mapsto f(a) \\ (2,b) \mapsto g(b) \end{pmatrix} \end{cases}$

Definition (Partial application of a bifunctor)**.** Let $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a bifunctor. Let $A \in \mathrm{ob}(\mathcal{C})$ and $X \in \mathrm{ob}(\mathcal{D})$. We write:

$$\begin{aligned} F(A,-)\colon Y \in \mathrm{ob}(\mathcal{D}) &\mapsto F(A,Y) \in \mathrm{ob}(\mathcal{E}) \\ f \in \mathcal{D}(Y,Z) &\mapsto F(A,f) = F(\mathrm{id}_A, f) \in \mathcal{E}(F(A,Y), F(A,Z)) \\ F(-,X)\colon B \in \mathrm{ob}(\mathcal{C}) &\mapsto F(B,X) \in \mathrm{ob}(\mathcal{E}) \\ g \in \mathcal{C}(B,C) &\mapsto F(g,X) = F(g, \mathrm{id}_X) \in \mathcal{E}(F(B,X), F(C,X)) \end{aligned}$$

One can check that these are functors: $F(A,-) \in [\mathcal{D},\mathcal{E}]$ and $F(-,X) \in [\mathcal{C},\mathcal{E}]$. For example, both the covariant hom-functor $\mathcal{C}(A,-)$ and the contravariant hom-functor $\mathcal{C}(-,X)$ are partial applications of the hom-bifunctor $\mathcal{C}(-,-)$.

PRODUCTS AND COPRODUCTS

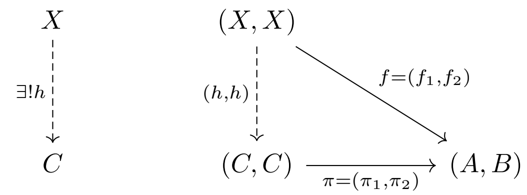
Idea: generalise the bifunctors $\times$ and $+$ on **Set** to other categories.

Definition. Let $\mathcal{C}$ be a category. We write $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ for its *diagonal functor*.

$$\Delta\colon A \in \mathrm{ob}(\mathcal{C}) \mapsto (A, A)$$
$$f \in \mathcal{C}(A, B) \mapsto (f, f)$$

Let A and B be two objects of $\mathcal{C}$. A *cartesian product* (or just *product*) of A and B is a universal morphism from Δ to $(A, B) \in \mathrm{ob}(\mathcal{C} \times \mathcal{C})$.

(*Important:* for now we only speak of *a* product, not *the* product.)

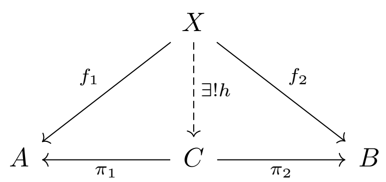


By definition, a universal morphism from Δ to (A, B) is pair (C, π) with $C \in \mathrm{ob}(\mathcal{C})$ and $\pi \in (\mathcal{C} \times \mathcal{C})((C, C), (A, B))$ that satisfies the universal property of the above diagram (for all X and f, there exists a unique h ...). We can write $f = (f_1, f_2)$ and $\pi = (\pi_1, \pi_2)$ because these are morphisms in the product category $\mathcal{C} \times \mathcal{C}$. Then:

$$f = \pi \circ (h, h) \quad \iff \quad f_1 = \pi_1 \circ h \text{ and } f_2 = \pi_2 \circ h$$

Therefore we can rephrase the definition more concretely: (C, π_1, π_2) is a product of A and B if and only if

for all $X \in \mathrm{ob}(\mathcal{C})$, $f_1 \in \mathcal{C}(X, A)$ and $f_2 \in \mathcal{C}(X, B)$, there exists a unique $h \in \mathcal{C}(X, C)$ making the diagram below commute:



Definition. In this case π_1 and π_2 are called the *projections* of this product; we also say that "C is a product of A and B with the projections π_1, π_2". We call h the *pairing* of f_1 and f_2 — notation: $h = \langle f_1, f_2 \rangle$.

Important: strictly speaking the data of the product is the whole triple (C, π_1, π_2)!

Examples:

In Set: $(A \times B,\ \pi_1,\ \pi_2)$ is a cartesian product of A and B with $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. Indeed, for $f_1\colon X \to A$, $f_2\colon X \to B$ and $h\colon X \to A \times B$, there is a unique $h = \langle f_1, f_2 \rangle$ that satisfies:

$$f_1 = \pi_1 \circ h \text{ and } f_2 = \pi_2 \circ h \quad \text{i.e.} \quad \forall x \in C,\ h(x) = (f_1(x), f_2(x))$$

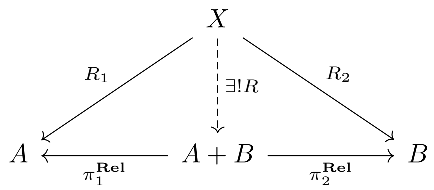
In Rel: $A + B = \{1\} \times A \cup \{2\} \times B$ is a product of A and B with the projections

$$\pi_1^{\mathbf{Rel}} = \{((1, a), a) \mid a \in A\} \in \mathbf{Rel}(A + B,\ A)$$
$$\pi_2^{\mathbf{Rel}} = \{((2, b), b) \mid b \in B\} \in \mathbf{Rel}(A + B,\ B)$$

because $R_1 = \pi_1^{\mathbf{Rel}} \circ R$ and $R_2 = \pi_2^{\mathbf{Rel}} \circ R$ if and only if

$$R = \{(x, (1, a)) \mid (x, a) \in R_1\} \cup \{(x, (2, b)) \mid (x, b) \in R_2\}$$

so this is the definition of the pairing $\langle R_1, R_2\rangle$ of two relations.

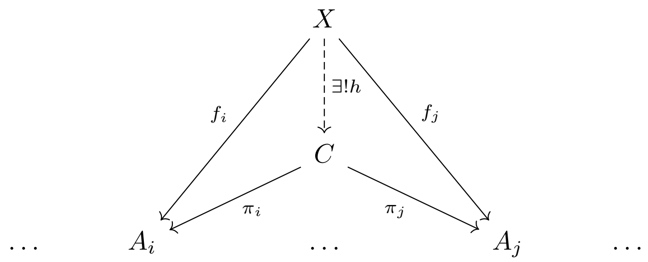


We can generalize products to an arbitrary family of objects — the previous case corresponds to a family of 2 objects.

Definition. Let I be a set and $(A_i)_{i\in I} \in \mathrm{ob}(\mathcal{C})^I$. A *product* $(C, (\pi_i)_{i\in I})$ of this family of objects consists of $C \in \mathrm{ob}(\mathcal{C})$ and $\pi_i \in \mathcal{C}(C, A_i)$ for $i \in I$ that satisfy the following equivalent conditions:

abstract: $(C, (\pi_i)_{i\in I})$ is a universal morphism from $\Delta\colon \mathcal{C} \to \mathcal{C}^I$ to $(A_i)_{i\in I}$ (where the power category $\mathcal{C}^I$ and the diagonal functor Δ are the expected generalisations of the binary case seen previously)

concrete: For every object X of $\mathcal{C}$ and family of morphisms $(f_i \in \mathcal{C}(X, A_i))_{i\in I}$, there exists a unique $h \in \mathcal{C}(X, C)$ such that $\pi_i \circ h = f_i$ for every $i \in I$.



The former examples generalize: in **Set**, the usual product $\prod_{i\in I} A_i$ of sets is a product of $(A_i)_{i\in I}$ with the projections $\pi_i =$ "select the i-th coordinate" for $i \in I$; in **Rel**, a product of $(A_i)_{i\in I}$ is given by the *dependent sum*

$$\sum_{i\in I} A_i = \{(i, a) \mid i \in I,\ a \in A_i\}$$

with the projections $\pi_i = \{((i, a), a) \mid a \in A_i\}$. Other examples include:

In Mon: If $(M_i)_{i\in I}$ is a family of monoids, the product $\prod_{i\in I} M_i$ of their underlying sets endowed with the monoid structure

$$(m_i)_{i\in I} \cdot (n_i)_{i\in I} = (m_i \cdot n_i)_{i\in I} \qquad e_{\prod_{i\in I} M_i} = (e_{M_i})_{i\in I}$$

is a product of this family in **Mon**, with the same projections as in **Set** (with this monoid structure, the maps π_i are monoid homomorphisms).

In PreOrd and Ord: Similarly to what happens with monoids, a product of $\big((X_i, \leqslant_i)\big)_{i\in I}$ is given by $\big(\prod_i X_i,\ \leqslant\big)$ for a certain (pre)order $\leqslant$, namely:

$$(x_i)_{i\in I} \leqslant (y_i)_{i\in I} \quad\iff\quad \forall i \in I,\ x_i \leqslant_i y_i$$

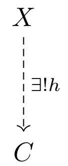
In $\mathcal{C}_{(X,\leqslant)}$ for a preordered set $(X, \leqslant)$: In this category there is at most one morphism of each "type" (source object + target object) so the commutativity of the diagram is irrelevant. What's important is the *existence* of morphisms: recall that $\mathcal{C}_{(X,\leqslant)}(x, y) \neq \varnothing$ if and only if $x \leqslant y$. When c is the product of $(a_i)_{i\in I} \in X^I$, the diagram tells us that:

- $\forall i \in I,\ c \leqslant a_i$ — in other words, c is a lower bound of $\{a_i \mid i \in I\}$
- any other lower bound x is below c.

Thus, c is a product of $(a_i)_{i\in I}$ if and only if it is a *greatest lower bound*, also called an *infimum*, of $\{a_i \mid i \in I\}$.

Remark. In a partially ordered set, the infimum is unique if it exists and one can write $c = \inf_{i\in I} a_i$. But in a preordered set this is not necessarily so: in $\mathbb{Z}$ equipped with the divisibility preorder, -1 and 1 are both infima of the whole set $\mathbb{Z}$.

In the special case $I = \varnothing$, all the A_i and all morphisms pointing to them disappear, and the diagram becomes:



A product of the empty family is just an object C (there are no projections) such that: for all $X \in \mathrm{ob}(\mathcal{C})$, there exists a unique $h \in \mathcal{C}(X, C)$... and that's it (there is no "such that", h is not required to make anything commute)! Thus, a 0-ary product is the same thing as a *terminal object*:

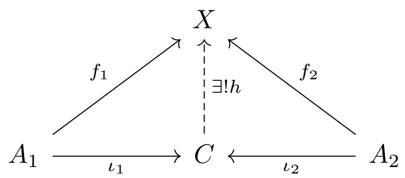
Definition. An object C is *terminal* in the category $\mathcal{C}$ when $\mathrm{card}(\mathcal{C}(X, C)) = 1$ for every $X \in \mathrm{ob}(\mathcal{C})$.

In Set, Mon and PreOrd: Singleton sets (equipped with the only possible structure) are terminal.
In Rel: The empty set is terminal.
In $\mathcal{C}_{(X,\leqslant)}$: Terminal objects are maximum elements of X.
(Indeed, an infimum of the empty subset is a maximum of the whole set!)

Now, every definition in category theory comes with a *dual* when applied to the opposite category.

Definition. A *coproduct* of a family of objects $(A_i)_{i\in I}$ in $\mathcal{C}^{\mathrm{op}}$ is a product of this family in $\mathcal{C}^{\mathrm{op}}$. Equivalently, it is a universal morphism from $(A_i)_{i\in I}$ to the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C}^I$.

For binary coproducts ($I = \{1, 2\}$) the diagram is as follows — of course, it is the diagram of a product with the arrows reversed.



In **Set**, the disjoint sum $A_1 + A_2$ is a coproduct of A_1 and A_2 with

$$\text{for } i \in \{1,2\}, \quad \iota_i\colon A_i \to A_1 + A_2$$
$$a \mapsto (i, a)$$

Indeed, the only map $h\colon A_1 + A_2 \to X$ that makes the diagram commute is

$$h\colon (i, a) \mapsto f_i(a)$$