(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: LECTURE 3

19 SEPTEMBER 2024 — L. T. D. NGUYỄN

Last time: a functor $F\colon \mathcal{C} \to \mathcal{D}$ (i.e. $f \in [\mathcal{C}, \mathcal{D}]$) consists of

$$A \in \mathrm{ob}(\mathcal{C}) \mapsto F(A) \in \mathrm{ob}(\mathcal{C})$$
$$f \in \mathcal{C}(A,B) \mapsto F(f) \in \mathcal{C}(A,B)$$

preserving composition and identities — and therefore all commutative diagrams, as well as isomorphisms.

Examples of functors

The identity functor $\mathrm{Id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ defined as: $\begin{cases} A \in \mathrm{ob}(\mathcal{C}) \mapsto A \\ f \in \mathcal{C}(A,B) \mapsto f \end{cases}$

The forgetful functor from Mon to Set: $\begin{cases} (M, \cdot, e) \in \mathrm{ob}(\mathbf{Mon}) \mapsto M \\ f \in \mathbf{Mon}(M,N) \mapsto f \end{cases}$

The forgetful functor from PreOrd to Set: similarly, maps a preordered set to the underlying set (forgetting the preorder) and a morphism (monotone function) to itself

Free monoid: $\begin{cases} A \in \mathrm{ob}(\mathbf{Set}) \mapsto A^* \in \mathrm{ob}(\mathbf{Mon}) \\ f \in \mathbf{Set}(A,B) \mapsto \big(f^*\colon [a_1, \dots, a_n] \mapsto [f(a_1), \dots, f(a_n)]\big) \end{cases}$

(In functional programming, f^* is called `map` f.)

Definition. An *endofunctor* of $\mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathcal{C}$.

Many *generic data structures* are examples of endofunctors on **Set** (this explains the usefulness of the `Functor` typeclass in Haskell):

List functor: $\begin{cases} A \in \mathrm{ob}(\mathbf{Set}) \mapsto A^* \in \mathrm{ob}(\overbrace{\mathbf{Set}}^{\text{main difference with free monoid functor}}) \\ f \in \mathbf{Set}(A,B) \mapsto f^* \in \mathbf{Set}(A^*, B^*) \end{cases}$

Option functor: Let $\mathsf{Option}(X) = \{\mathsf{Some}(x) \mid x \in X\} \cup \{\mathsf{None}\}$, just like the `'a option` data type in OCaml (called `Maybe a` in Haskell).

$$\begin{cases} A \in \mathrm{ob}(\mathbf{Set}) \mapsto \mathsf{Option}(A) \in \mathrm{ob}(\mathbf{Set}) \\ f \in \mathbf{Set}(A,B) \mapsto \begin{pmatrix} \mathsf{Option}(f)\colon \mathsf{Some}(a) \mapsto \mathsf{Some}(f(a)) \\ \mathsf{None} \mapsto \mathsf{None} \end{pmatrix} \end{cases}$$

Pair functor: $\begin{cases} A \in \mathrm{ob}(\mathbf{Set}) \mapsto A^2 \in \mathrm{ob}(\mathbf{Set}) \\ f \in \mathbf{Set}(A,B) \mapsto \big((x,y) \mapsto (f(x), f(y))\big) \end{cases}$

Noting that $A^2 \cong \mathbf{Set}(\{1,2\}, A)$, the last example is “the same” as the functor $\mathbf{Set}(\{1,2\}, -)$ defined below. (The rigorous definition of “the same” is *naturally isomorphic*; this will be defined later.)

Proposition (*important!*)**.** *Let $\mathcal{C}$ be a locally small category and $X \in \mathrm{ob}(\mathcal{C})$.*
The following defines a functor $\mathcal{C}(X, -)$ (called "covariant Hom-functor"*):*

$$\begin{aligned} \mathcal{C}(X, -)\colon \mathcal{C} &\to \mathbf{Set} \\ A \in \mathrm{ob}(\mathcal{C}) &\mapsto \mathcal{C}(X, A) \in \mathrm{ob}(\mathbf{Set}) \\ f \in \mathcal{C}(A, B) &\mapsto \begin{pmatrix} \mathcal{C}(X, f)\colon \mathcal{C}(X, A) \to \mathcal{C}(X, B) \\ g \mapsto f \circ g \end{pmatrix} \end{aligned}$$

Remark (Non-examples)**.** There is no "obvious" way to extend these operations on sets into endofunctors on **Set**:

- $A \in \mathrm{ob}(\mathbf{Set}) \mapsto A^A$ i.e. $\mathbf{Set}(A, A)$
- $A \in \mathrm{ob}(\mathbf{Set}) \mapsto \{(x, y) \in A^2 \mid x \neq y\}$

The following does *not* define a functor $\mathbf{Set} \to \mathbf{Set}$ (why?):

$$\begin{aligned} A \in \mathrm{ob}(\mathbf{Set}) &\mapsto \mathsf{Option}(A) \\ f \in \mathbf{Set}(A, B) &\mapsto (x \mapsto \mathsf{None}) \end{aligned}$$

We can also look at functors on our standard examples of small categories:

Proposition (Functors generalise monoid homomorphisms & montone functions)**.** *For M and N monoids, the following is a well-defined bijection (with $\mathrm{ob}(\mathcal{C}_M) = \{*\}$):*

$$\begin{aligned} [\mathcal{C}_M, \mathcal{C}_N] &\to \mathbf{Mon}(M, N) \\ F &\mapsto \begin{pmatrix} M \to N \\ x \mapsto F(x \textit{ seen as a morphism } * \to *) \end{pmatrix} \end{aligned}$$

For $(X, \leqslant)$ and $(Y, \leqslant)$ preordered sets, the following is a well-defined bijection:

$$\begin{aligned} [\mathcal{C}_{(X,\leqslant)}, \mathcal{C}_{(Y,\leqslant)}] &\to \mathbf{PreOrd}((X, \leqslant), (Y, \leqslant)) \\ F &\mapsto \begin{pmatrix} X \to Y \\ x \mapsto F(x \textit{ seen as an object of } \mathcal{C}_{(X,\leqslant)}) \end{pmatrix} \end{aligned}$$

A final example: for $\mathcal{G}$ the path category on some graph,

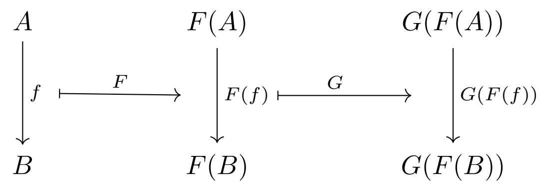
$$\begin{aligned} \mathcal{G} &\to \mathcal{C}_{(\mathbb{N},+,0)} \\ \text{(vertex) } u \in \mathrm{ob}(\mathcal{G}) &\mapsto * \in \mathrm{ob}(\mathcal{C}_{(\mathbb{N},+,0)}) \\ \text{(path) } p \in \mathcal{G}(u, v) &\mapsto (\text{length of } p) \in \mathbb{N} = \mathcal{C}_{(\mathbb{N},+,0)}(*, *) \end{aligned}$$

is a functor because the length of the concatenation of two paths is the sum of the lengths of these two paths.

Composing functors

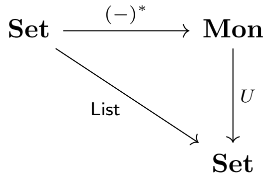
Definition. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors. We define $G \circ F$ as:

$$\begin{aligned} A \in \mathrm{ob}(\mathcal{C}) &\mapsto G(F(A)) \in \mathrm{ob}(\mathcal{E}) \\ f \in \mathcal{C}(A, B) &\mapsto G(F(f)) \in \mathcal{E}(G(F(A)), G(F(B))) \end{aligned}$$



Proposition. *$G \circ F$ is a functor (it preserves $\circ$ and* id*).*
Furthermore, this composition is associative with the identity functors as units.

This allows us to reason on compositions of functors by drawing commutative diagrams. For instance List $= U \circ (-)^*$, where $U\colon \mathbf{Mon} \to \mathbf{Set}$ is the forgetful functor, can be represented as:



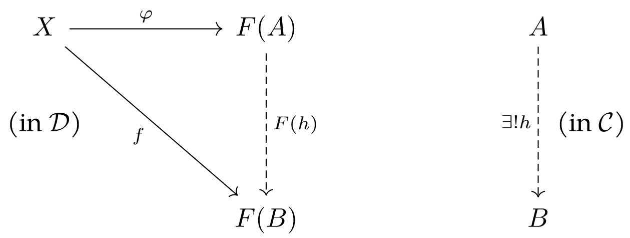
Remark. It is tempting to speak of the "category of categories" (morphisms = functors, thus endomorphisms = endofunctors ...). This poses some foundational issues (*à la* set of all sets) — instead, one can consider the category of *small* categories, or use a hierarchy of universes.[1] Anyway, the "right point of view" would be to consider the *2-category of categories*; 2-categories are out of scope for us.

Using functors to state universal properties

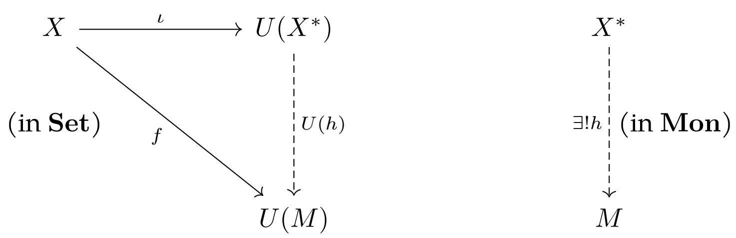
We introduce a general definition that covers many cases of universal properties.

Definition. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and $X \in \mathrm{ob}(\mathcal{D})$. A pair (A, φ) where $A \in \mathrm{ob}(\mathcal{C})$ and $\varphi \in \mathcal{D}(X, F(A))$ is a *universal morphism from X to F* when:

$$\forall B \in \mathrm{ob}(\mathcal{C}),\ \forall f \in \mathcal{D}(X, F(B)),\ \exists! h \in \mathcal{C}(A, B) : f = F(h) \circ \varphi$$



Example: The universal property of the free monoid (cf. Lecture 1) says that (X^*, ι), where $\iota(x) = [x]$, is a universal morphism from $X \in \mathrm{ob}(\mathbf{Set})$ to the forgetful functor $U\colon \mathbf{Mon} \to \mathbf{Set}$



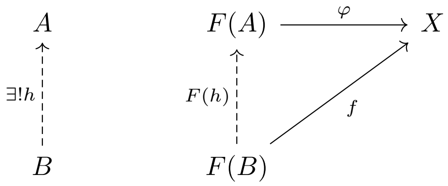
Indeed, $U(X^*) = X^*$, $U(M) = M$ and $U(h) = h$; moreover, a function is of the form $U(h)$ for $h \in \mathbf{Mon}(X^*, M)$ if and only if it is a monoid homomorphism $X^* \to M$, so we recover the "$\exists! h$ homomorphism" of Lecture 1!

There is a version of universal morphisms where "the arrows go the other way":

Definition. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and $X \in \mathrm{ob}(\mathcal{D})$. A pair (A, φ) where $A \in \mathrm{ob}(\mathcal{C})$ and $\varphi \in \mathcal{D}(F(A), X)$ is a *universal morphism from F to X* when:

$$\forall B \in \mathrm{ob}(\mathcal{C}),\ \forall f \in \mathcal{D}(F(B), X),\ \exists! h \in \mathcal{C}(B, A) : f = \varphi \circ F(h)$$

[1] Usually called "Grothendieck universes" [nLa24] but already considered in the first half of the 20th century by set theorists [Ham22]. Universe levels in Coq are a related concept.



Example: the universal property of the trivial preorder $\longrightarrow$ see Homework 1.

Duality

Many definitions in category theory come in two variants, with the "direction of the arrows" mirrored. As we just saw, this is the case of universal morphisms. Let us formalise this duality by introducing the operation that reverses all the arrows (morphisms) in a category.

Definition. Let $\mathcal{C}$ be a category. Its *opposite category* $\mathcal{C}^{\mathrm{op}}$ has:

- $\mathrm{ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{ob}(\mathcal{C})$
- $\mathcal{C}^{\mathrm{op}}(A, B) = \mathcal{C}(B, A)$ for $A, B \in \mathrm{ob}(\mathcal{C}^{\mathrm{op}})$
- $f \circ_{\mathcal{C}^{\mathrm{op}}} g = g \circ_{\mathcal{C}} f$
- id_A in $\mathcal{C}^{\mathrm{op}} = \mathrm{id}_A$ in $\mathcal{C}$

$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \qquad \text{in } \mathcal{C}$$

$$A \xleftarrow[\;f\;]{} B \xleftarrow[\;g\;]{} C \qquad \text{in } \mathcal{C}^{\mathrm{op}}$$

One can check from the definitions that $\mathcal{C}^{\mathrm{op}}$ is indeed a category (that is, $\circ_{\mathcal{C}^{\mathrm{op}}}$ is associative, with unit id).

Proposition. *$(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ for any category $\mathcal{C}$.*

Remark. This means that $\mathcal{C} = \mathcal{D}^{\mathrm{op}} \iff \mathcal{C}^{\mathrm{op}} = \mathcal{D}$. We then say that the categories $\mathcal{C}$ and $\mathcal{D}$ are *dual*, which is a *symmetric* relation.

The operation $(-)^{\mathrm{op}}$ also applies to functors:

Proposition. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. The following is also a functor:*

$$\begin{aligned} F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} &\to \mathcal{D}^{\mathrm{op}} \\ A &\mapsto F(A) \\ f &\mapsto F(f) \end{aligned}$$

Basically F^{op} does the same thing as F but this is interpreted differently because of the $(-)^{\mathrm{op}}$ on the domain and codomain. Thanks to this definition, we can show that the two notions of universal morphisms are redundant: they are *dual*.

Proposition. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and $X \in \mathrm{ob}(\mathcal{D})$.*

universal morphism from X to F^{op} = universal morphism from F to X
universal morphism from F^{op} to X = universal morphism from X to F

Next, we look at an important functor involving $(-)^{\mathrm{op}}$.

The contravariant Hom functor: Let $\mathcal{C}$ be a locally small category and X be an object of $\mathcal{C}$ — equivalently, of $\mathcal{C}^{\mathrm{op}}$. According to an earlier proposition, we can form the functor $\mathcal{C}^{\mathrm{op}}(X, -)$. It is also denoted by $\mathcal{C}(-, X)$ and an explicit description in terms of $\mathcal{C}$ is:

$$\begin{aligned} \mathcal{C}^{\mathrm{op}} &\to \mathbf{Set} \\ A \in \mathrm{ob}(\mathcal{C}) &\mapsto \mathcal{C}(A, X) \in \mathrm{ob}(\mathbf{Set}) \\ f \in \mathcal{C}(B, A) &\mapsto \begin{pmatrix} \mathcal{C}(f, X) \colon \mathcal{C}(A, X) \to \mathcal{C}(B, X) \\ \hfill g \mapsto g \circ f \end{pmatrix} \end{aligned}$$

Why "contravariant Hom functor"? Because we say in general that:

Definition. A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Remark. A functor $\mathcal{C} \to \mathcal{D}$ without the $(-)^{\mathrm{op}}$ is also called sometimes a "covariant functor" from $\mathcal{C}$ to $\mathcal{D}$.

A contravariant functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ "reverses composition" in the sense that

$$F(f \circ_{\mathcal{C}} g) = F(g \circ_{\mathcal{C}^{\mathrm{op}}} f) = F(g) \circ_{\mathcal{D}} F(f)$$

We have seen that the operation $A, B \mapsto \mathcal{C}(A, B)$ is somehow "contravariant in the first argument" ($\mathcal{C}(-, B) \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$) while it is "covariant in the second argument" ($\mathcal{C}(A, -) \colon \mathcal{C} \to \mathcal{D}$). Let us make it "functorial" in *both arguments at the same time*.

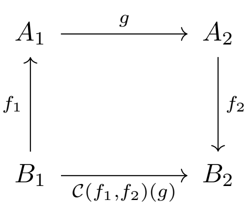
Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Their *product category* $\mathcal{C} \times \mathcal{D}$ is:

- $\mathrm{ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$
- $(\mathcal{C} \times \mathcal{D})((A_1, A_2), (B_1, B_2)) = \mathcal{C}(A_1, B_1) \times \mathcal{D}(A_2, B_2)$

(A short verification shows that it is indeed a category.)

Proposition (Hom-bifunctor)**.** *Let $\mathcal{C}$ be a locally small category. The following defines a functor $\mathcal{C}(-, -) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$.*

$$\begin{aligned} (A_1, A_2) &\mapsto \mathcal{C}(A_1, A_2) \\ (f_1, f_2) \in \underbrace{\mathcal{C}(B_1, A_1) \times \mathcal{C}(A_2, B_2)}_{= (\mathcal{C}^{\mathrm{op}} \times \mathcal{C})((A_1, A_2), (B_1, B_2))} &\mapsto \begin{pmatrix} \mathcal{C}(f_1, f_2) \colon \mathcal{C}(A_1, A_2) \to \mathcal{C}(B_1, B_2) \\ \hfill g \mapsto f_2 \circ g \circ f_1 \end{pmatrix} \end{aligned}$$



References

[Ham22] Joel David Hamkins. Authorship of Grothendieck universes (MathOverflow answer), 2022. URL: `https://mathoverflow.net/q/433376`.

[nLa24] nLab authors. Grothendieck universe. `https://ncatlab.org/nlab/show/Grothendieck+universe`, September 2024. Revision 58.