(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: LECTURE 2

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Another example of structure: (pre)orders

Definition. A *preorder* on a set X is a binary relation $(\leqslant) \subseteq X \times X$ which is:

reflexive: $\forall x \in X,\ x \leqslant x$

transitive: $\forall x, y, z \in X,\ (x \leqslant y)$ and $(y \leqslant z) \implies x \leqslant z$

Furthermore, $\leqslant$ is a *partial order* if it is a preorder that is also

antisymmetric: $\forall x, y \in X,\ (x \leqslant y)$ and $(y \leqslant x) \implies x = y$

When $\leqslant$ is a preorder (resp. partial order) on X, the pair $(X, \leqslant)$ is called a *preordered set* (resp. *partially ordered set*, often abbreviated as "poset").

Examples of posets include:

- $(\mathbb{N}, \leqslant)$ or $(\mathbb{R}, \leqslant)$ with the usual order
- the powerset $\mathcal{P}(A) = \{X \mid X \subseteq A\}$ with the inclusion relation

An important source of preordered sets is:

Proposition. *Let $(M, \cdot, e)$ be a monoid. Then the following defines a preorder $\preceq$ on M:*

$$x \preceq y \quad \text{when} \quad \exists z \in M : x \cdot z = y$$

(*which may be called "left divisibility"*).

- over $(\mathbb{N}, +, 0)$ it defines the usual order
- over $(\mathbb{Z}, +, 0)$ it is the *trivial preorder*: $x \preceq y$ for *all* $x, y \in \mathbb{Z}$ — definitely not antisymmetric!
- over $(\mathbb{N}, \times, 1)$ or $(\mathbb{Z}, \times, 1)$ it is the *divisibility* relation — antisymmetric over $\mathbb{N}$ but not over $\mathbb{Z}$
- over the free monoid $(X^*, \cdot, [])$ it is the "prefix" relation

(Cultural remark: more generally this preorder is one of "Green's relations" on monoids and plays an important role in automata theory, see e.g. [Boj20].)

Definition. Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be preordered sets. A function $f\colon X \to Y$ is *monotone* when $\forall x, y \in X,\ x \leqslant_X y \implies f(x) \leqslant_Y f(y)$.

For example, $\ell \in X^* \mapsto$ (set of elements appearing in ℓ) $\in \mathcal{P}(X)$ is monotone.

Proposition. *Any monoid homomorphism $h\colon M \to N$ is monotone from $(M, \preceq)$ to $(N, \preceq)$ (using the above-defined left divisibility preorder).*

Proof idea. Apply h to the equation $x \cdot y = z$ and use the homomorphism property. □

Categories: basic definitions (continued)

Recall that a category $\mathcal{C}$ consists of a collection of *objects* $\mathrm{ob}(\mathcal{C})$ and of collections of *morphisms* $\mathcal{C}(A, B)$ for any $A, B \in \mathrm{ob}(\mathcal{C})$, endowed with an associative composition of morphisms and an identity morphism on each object. We saw the examples **Set** (sets and functions), **Mon** (monoids and homomorphisms) and **Rel** (sets and binary relations). Two examples similar to **Set** and **Mon** are:

the category of preorders: $\mathrm{ob}(\mathbf{PreOrd})$ = preordered sets,
$\mathbf{PreOrd}(A, B)$ = monotone functions from A to B

the category of posets: $\mathrm{ob}(\mathbf{Ord})$ = partially ordered sets,
$\mathbf{Ord}(A, B)$ = monotone functions from A to B

These are categories because the composition of two monotone functions is also monotone, and the identity function is monotone.

We also discussed set-theoretic "size issues" — the fact that the collections of all sets, of all monoids, etc. are not sets. There is some vocabulary for that:

Definition. A category $\mathcal{C}$ is:

- *locally small* when $\mathcal{C}(A, B)$ is a set for all $A, B \in \mathrm{ob}(\mathcal{C})$
- *small* when it is locally small and $\mathrm{ob}(\mathcal{C})$ is a set

In general, the categories of "sets with structure" **Set**, **Mon**, **Rel**, **PreOrd**, etc. tend to be locally small but not small. *Nearly all our examples of categories in this course will be locally small.*

Examples of small categories include:

The empty category: its collection of objects is $\varnothing$

The smallest non-empty category: $\mathrm{ob}(\mathcal{C}) = \{A\}$, $\mathcal{C}(A, A) = \{\mathrm{id}_A\}$

The path category on a graph: Fix a directed graph. From this graph one can build a category $\mathcal{G}$ with

- $\mathrm{ob}(\mathcal{G})$ = vertices of the graph
- $\mathcal{G}(u, v)$ = paths from u to v
- composition is path concatenation, id_u is the empty path

The objects in this category $\mathcal{G}$ are *not* "structured sets", we are not supposed to talk about their "elements". In other words:

for A an object of a category, "$a \in A$" does not always make sense.

Here are some other important constructions of small categories whose objects are not set-like:

From a monoid to a category: Let $(M, \cdot, e)$ be a monoid. We can define the category $\mathcal{C}_M$ as:

- there is a single object, which is arbitrary, let's call it $*$
- $\mathcal{C}(*, *) = M$
- $x \circ y = x \cdot y$ for $x, y \in \mathcal{C}(*, *)$
- $\mathrm{id}_* = e$

Associativity/unitality in M implies associativity/unitality in $\mathcal{C}_M$.
$\longrightarrow$ slogan: "*categories generalize monoids* by allowing composition to be defined only in 'well-typed' situations (types = objects)"

From a preorder to a category: Let $(X, \leqslant)$ be a preordered set. We can define the category $\mathcal{C}_{(X,\leqslant)}$ (or $\mathcal{C}_X$ slightly abusively) as:

- $\mathrm{ob}(\mathcal{C}_{(X,\leqslant)}) = X$
- $\mathcal{C}_{(X,\leqslant)}(x, y) = \begin{cases} \{*\} & \text{if } x \leqslant y \\ \varnothing & \text{otherwise} \end{cases}$

where $*$ is arbitrary. There is only one possible way to define composition and identities. Note that

$$\text{existence of composition} = \text{transitivity of } \leqslant$$
$$\text{existence of identities} = \text{reflexivity of } \leqslant$$

and associativity/unitality are trivial: $\mathcal{C}(x, y)$ contains at most one element, so they must all be equal.
$\longrightarrow$ slogan: "*categories generalize preorders* by replacing a truth value $A \leqslant B$ with a collection of 'witnesses' $\mathcal{C}(A, B)$".

Proposition. *Conversely, for a category $\mathcal{C}$:*

- *if $\mathcal{C}$ is locally small and $X \in \mathrm{ob}(\mathcal{C})$, then $(\mathcal{C}(X,X), \circ, \mathrm{id}_X)$ is a monoid – the monoid of* endomorphisms *of X;*
- *if $\mathcal{C}$ is small, then $(\mathrm{ob}(\mathcal{C}), \leqslant)$ is a preordered set, where $A \leqslant B$ is defined as $\mathcal{C}(A,B) \neq \varnothing$.*

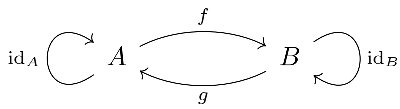
For example:

- The monotone functions from a preordered set to itself form a monoid.
- The preorder coming from the path category of a directed graph is the *reachability* relation $u \leqslant v \iff$ there exists a path from u to v.

Isomorphisms

Definition. Let $\mathcal{C}$ be a category and $A, B \in \mathrm{ob}(\mathcal{C})$. A morphism $f \in \mathcal{C}(A,B)$ is an *isomorphism* when there exists $g \in \mathcal{C}(B,A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Remark. This amounts to saying that the following diagram commutes:



Proposition. *In that case, g is unique; we call it the* inverse *of f and denote it by f^{-1}.*

Proof. If g, g' are two inverses then $g = g \circ \mathrm{id}_B = g \circ f \circ g' = \mathrm{id}_A \circ g' = g'$. □

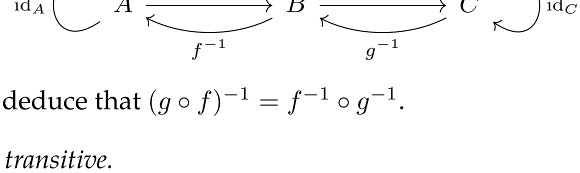
Definition. We write $\mathrm{Iso}_{\mathcal{C}}(A,B)$ for the collections of isomorphisms from A to B. When $\mathrm{Iso}_{\mathcal{C}}(A,B) \neq \varnothing$ we say that A and B are isomorphic (notation: $A \cong B$).

In our examples of categories:

In Set: bijections
In Rel: relations $\{(a, f(a)) \mid a \in A\} \subseteq A \times B$ where $f\colon A \to B$ is a bijection
In Mon: monoid isomorphisms = bijective homomorphisms (cf. Lecture 1)
In PreOrd and Ord: strictly included in bijective monotone functions!
For instance: the bijection $\mathrm{id}_{\mathbb{N}}$ is monotone from $(\mathbb{N}, =)$ to $(\mathbb{N}, \leqslant)$; if it had an inverse, it would necessarily be its inverse bijection (itself), which is not monotone from $(\mathbb{N}, \leqslant)$ to $(\mathbb{N}, =)$, so there is no inverse.
In a category coming from a preordered set $(X, \leqslant)$: we have $x \cong y$ in $\mathcal{C}_{(X,\leqslant)}$ if and only if $\underbrace{x \leqslant y \text{ and } y \leqslant x}_{\text{equivalence relation induced by the preorder } \leqslant}$.

Proposition. *The composition of two isomorphisms is an isomorphism.*

Proof idea. The diagram below commutes because its restrictions to $\{A, B\}$ and to $\{B, C\}$ commute:



from which we deduce that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

Corollary. *$\cong$ is transitive.*

Functors

Idea: functors are "morphisms between categories".

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of:

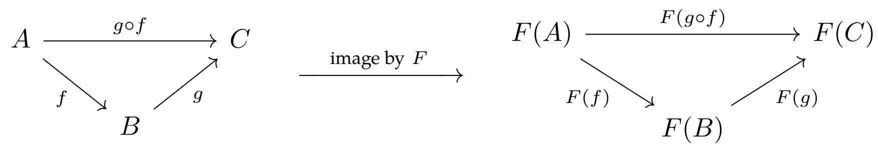
- for every $A \in \mathrm{ob}(\mathcal{C})$, an object $F(A) \in \mathrm{ob}(\mathcal{D})$
- for all $A, B \in \mathrm{ob}(\mathcal{C})$ and $f \in \mathcal{C}(A, B)$, a morphism $\underbrace{F(f)}_{\text{abuse of notation: also depends on } A \text{ and } B\text{, not just } f} \in \mathcal{D}(F(A), F(B))$

such that

- $\forall A \in \mathrm{ob}(\mathcal{C}),\ F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$
- $\forall A, B, C \in \mathrm{ob}(\mathcal{C}),\ \forall f \in \mathcal{C}(A, B),\ \forall g \in \mathcal{C}(B, C),\ F(g \circ f) = F(g) \circ F(f)$

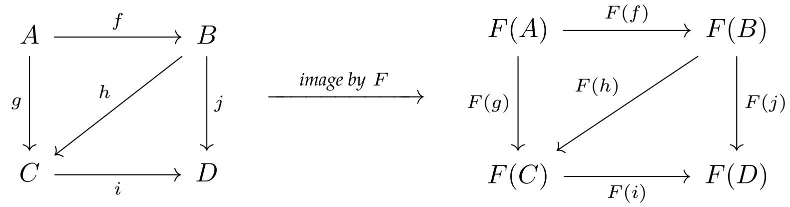
Notation: $[C, D]$ is the collection of all functors from $\mathcal{C}$ to $\mathcal{D}$.

Remark. Diagrammatically:



The diagram on the left commutes by definition of $\circ$. The commutation of the right diagram is the axiom $F(g \circ f) = F(g) \circ F(f)$.

Proposition. *A functor preserves* all *commutative diagrams, for instance*

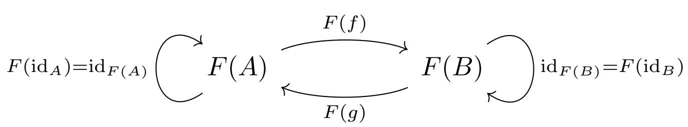


This is inconvenient to prove rigorously in full generality, as we have not defined formally what a commutative diagram is, but it makes intuitive sense. You may try to check, using the functor axioms, that the above example indeed works.

As an application:

Corollary. *For $f \in \mathrm{Iso}_{\mathcal{C}}(A, B)$ and $F\colon \mathcal{C} \to \mathcal{D}$ a functor, $F(f) \in \mathrm{Iso}_{\mathcal{D}}(F(A), F(B))$.*

Proof. Take the image by F of the diagram stating that f is an isomorphism:



We see that $F(f)^{-1} = F(f^{-1})$ as we would have expected! ☐

Next time we will see several examples of functors.

References

[Boj20] Mikołaj Bojańczyk. Languages recognised by finite semigroups, and their generalisations to objects such as trees and graphs, with an emphasis on definability in monadic second-order logic, 2020. Online book. `arXiv:2008.11635`.