(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: HOMEWORK 4

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In this homework, we fix a monoid $(M, \cdot, e)$. An *M-action* $(X, \bullet)$ consists of a set X and a binary operation $\bullet\colon X \times M \to X$ such that

$$\forall x \in X,\ \forall m, n \in M,\ x \bullet (m \cdot n) = (x \bullet m) \bullet n \text{ and } x \bullet e = x$$

(For instance, deterministic finite automata over an alphabet Σ can be seen as Σ^*-actions on a set of states.) A *morphism of M-actions* from $(X, \bullet_X)$ to $(Y, \bullet_Y)$ is a function $f\colon X \to Y$ such that

$$\forall x \in X,\ \forall m \in M,\ f(x \bullet_X m) = f(x) \bullet_Y m$$

We admit that M-actions and their morphisms form a category $\mathbf{Act}[M]$ with the usual function composition and identities (from $\mathbf{Set}$). It is then immediate that there is a forgetful functor $U\colon \mathbf{Act}[M] \to \mathbf{Set}$ that maps $(X, \bullet)$ to X and a morphism to itself. We also consider the functor:

$$\begin{aligned} F\colon \mathbf{Set} &\to \mathbf{Act}[M] \\ X &\mapsto X \times M \text{ with } (x, m) \bullet n = (x,\ m \cdot n) \\ (f\colon X \to Y) &\mapsto f \times \mathrm{id}_M \text{ (using the bifunctor } (- \times -) \text{ on } \mathbf{Set}) \end{aligned}$$

1. Exhibit an adjunction $F \dashv U$ by giving a natural bijection between hom-sets. You are not required to prove naturality.

2. Show that the unit of this adjunction is $\eta_X\colon x \mapsto (x, e)$, and state the universal property of $(F(X), \eta_X)$.

3. Compute the counit $\varepsilon\colon F \circ U \Rightarrow \mathrm{Id}_{\mathbf{Act}[M]}$.

4. Let $W = U \circ F$ be the monad induced by this adjunction (it is called the *writer monad*). Compute its multiplication $\mu\colon W \circ W \Rightarrow W$.

5. Since W is a monad on $\mathbf{Set}$, we can consider the "bind" operator. Show that $(x, m) \gg\!\!= f = (\pi_1(f(x)),\ m \cdot \pi_2(f(x)))$.

6. We now take $(M, \cdot, e)$ to be the free monoid $(\mathbb{N}^*, \cdot, [])$ — note that $\mathbb{N}^*$ refers to the set of lists with elements in $\mathbb{N}$. Let $\texttt{print}(n) = (*, n) \in W(\{*\})$. Describe the result of the following computation:

$$\eta_{\mathbb{N}}(6) \gg\!\!= (x \mapsto \texttt{print}(x) \gg\!\!= (y \mapsto \texttt{print}(7 \times x)))$$

(No detailed justification necessary.)