(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: HOMEWORK 3

30 SEPTEMBER 2024 — L. T. D. NGUYỄN

Exercise 1

Let $(\mathcal{C}, \&, \top)$ be a cartesian category and $A, B, C \in \mathrm{ob}(\mathcal{C})$.

1. For each $X \in \{A, B, C\}$, define a morphism $p_X \in \mathcal{C}((A \,\&\, B) \,\&\, C,\ X)$ using projections and composition.

2. Let $Y \in \mathrm{ob}(\mathcal{C})$ and $h \in \mathcal{C}(Y,\ (A \,\&\, B) \,\&\, C)$. Let $f_A = p_A \circ h$ and likewise $f_B = p_B \circ h$. Prove that $\pi_1^{(A\&B),C} \circ h = \langle f_A, f_B \rangle$, where $\langle -, - \rangle$ is the pairing for some product — which one?

3. Show that $((A \,\&\, B) \,\&\, C,\ p_A,\ p_B,\ p_C)$ is a product of the family (A, B, C). *Hint: use the previous question as an intermediate step.*

4. Suppose that we also know that $A \,\&\, (B \,\&\, C)$ is a product of A, B, C (with suitable projections). What can we then conclude?

Exercise 2

We write $\mathcal{P}(A) = \{X \mid X \subseteq A\}$ for the powerset of the set A. For two sets A and B, a subset $X \in \mathcal{P}(A)$ and a binary relation $R \subseteq A \times B$, let

$$R_\dagger X = \{b \in B \mid \exists a \in X : (a, b) \in R\}$$

Consider the following endofunctor on the category $\mathbf{Rel}$ of relations (we admit that it is a functor, i.e. preserves composition and identities):

$$\begin{aligned} F \colon \mathbf{Rel} &\to \mathbf{Rel} \\ A \in \mathrm{ob}(\mathbf{Rel}) &\mapsto \mathcal{P}(A) \\ R \in \mathbf{Rel}(A, B) &\mapsto \{(X, R_\dagger X) \mid X \in \mathcal{P}(A)\} \end{aligned}$$

1. Show that the following family $(\alpha_A)_{A \in \mathrm{ob}(\mathbf{Rel})}$ of binary relations defines a natural transformation $\alpha \colon F \Rightarrow \mathrm{Id}_{\mathbf{Rel}}$: $\alpha_A = \{(X, x) \mid X \in \mathcal{P}(A),\ x \in X\}$.

2. Show that the following family $(\beta_A)_{A \in \mathrm{ob}(\mathbf{Rel})}$ of relations *does not* define a natural transformation from F to $\mathrm{Id}_{\mathbf{Rel}}$: $\beta_A = \{(\{a\}, a) \mid a \in A\}$.
Hint: look at the naturality condition for $\{(0, 1), (0, 2)\} \in \mathbf{Rel}(\{0\}, \{1, 2\})$.