(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: HOMEWORK 2

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Exercise 1

1. Show that the following defines a functor:

$$\begin{aligned} \mathrm{Rev}\colon \mathbf{Rel}^{\mathrm{op}} &\to \mathbf{Rel} \\ A &\mapsto A \text{ on objects} \\ R \subseteq B \times A &\mapsto \{(a,b) \mid (b,a) \in R\} \text{ on morphisms} \end{aligned}$$

2. What can you say about $\mathrm{Rev} \circ \mathrm{Rev}^{\mathrm{op}}$ and $\mathrm{Rev}^{\mathrm{op}} \circ \mathrm{Rev}$?

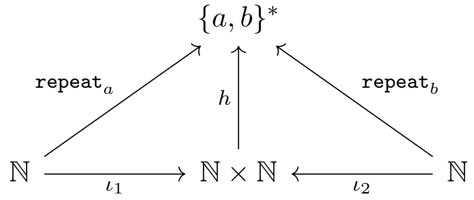
Exercise 2

A *commutative monoid* is a monoid M such that $x \cdot y = y \cdot x$ for all $x, y \in M$. Let **CMon** be the category whose objects are commutative monoids and whose morphisms are monoid homomorphisms, with usual composition and identities.

1. Let M and N be commutative monoids. Show that $(M \times N,\, \iota_1,\, \iota_2)$ is a coproduct *in the category* **CMon** of M and N, where $\iota_1\colon m \in M \mapsto (m, e_N)$ and $\iota_2\colon n \in N \mapsto (e_M, n)$. ($e_M$ is the unit element of M.)

Hint: given $f\colon M \to P$ and $g\colon N \to P$ two monoid homomorphisms, with P commutative, consider $(f \cdot g)\colon (m,n) \mapsto f(m) \cdot g(n)$.

2. We consider $\mathbb{N}$ as the monoid $(\mathbb{N}, +, 0)$. Let a and b be two different letters, $\texttt{repeat}_a\colon n \in \mathbb{N} \mapsto a \ldots a$ (n times) and $\texttt{repeat}_b\colon n \in \mathbb{N} \mapsto b \ldots b$ (n times). Show that *there does* **not** *exist any* $h \in \mathbf{Mon}(\mathbb{N} \times \mathbb{N}, \{a,b\}^*)$ such that the diagram below commutes, where ι_1, ι_2 are defined as in the previous question (with $e_{\mathbb{N}} = 0$):



Hint: use the *noncommutativity* of the monoid $\{a,b\}^*$.

3. What can you conclude about coproducts *in the category* **Mon**?