(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: HOMEWORK 1

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Exercise 1

1. Describe the small category $\mathcal{C} = \mathcal{C}_{(\{0,1\},\leqslant)}$ corresponding to $\{0,1\}$ endowed with its usual order.

2. Exhibit a functor $F\colon \mathcal{C} \to \mathbf{Set}$ (and justify that it is a functor) such that:

- $F(0) = F(1) = \mathbb{N}$,
- there exist $A, B \in \mathrm{ob}(\mathcal{C})$ and $f \in \mathcal{C}(A,B)$ such that $F(f)$ is the function $n \mapsto n + 42$.

(Important : $* \in \mathcal{C}(0,0)$ and $* \in \mathcal{C}(0,1)$ are considered to be different morphisms because their target object is not the same; the functor is allowed to treat them differently. This also applies when the source object differs.)

3. Show that there does *not* exist any morphism g such that $F(g) = F(f) \circ F(f)$. In particular, why *cannot* we say that "$F(f \circ f) = F(f) \circ F(f)$" even though F is a functor?

Exercise 2

Throughout this exercise, we fix some set X.

1. Let $(A_1, \leqslant_1)$ and $(A_2, \leqslant_2)$ be two preordered sets, and $\varphi_i\colon A_i \to X$ be a function for each $i \in \{1,2\}$. Suppose that the following universal property holds for both $i = 1$ and $i = 2$:

for every preordered set $(B, \leqslant)$ and function $f\colon B \to X$, there exists a unique monotone function $h\colon B \to A_i$ such that $f = \varphi_i \circ h$.

(Note that the functions with codomain X are not required to be monotone — that requirement would not make sense because we have not equipped X with a preorder.) Prove that $(A_1, \leqslant_1)$ and $(A_2, \leqslant_2)$ are isomorphic in $\mathbf{PreOrd}$.

(Hint: adapt a proof from Lecture 1.)

2. Let $\leqslant_{\mathrm{triv}}$ be the *trivial preorder* on the set X, that is, let $x \leqslant_{\mathrm{triv}} y$ be *always* true for $x, y \in X$. Show that $(X, \leqslant_{\mathrm{triv}})$, with the right choice of function φ, satisfies the universal property.

3. Can you express this as a universal morphism *from F to X* for some functor F seen in the lectures?

4. Give (without justification) a universal morphism *from X to F* for the same F.