The structure of polynomial growth for tree automata/transducers and MSO set queries

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joint work with Paul Gallot (Univ. Bremen) & Nathan Lhote (AMU)

Highlights of Logic, Games and Automata 2025 (Saarbrücken)

Overview

Basic results on finite automata (known + obvious extension)

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"Reparameterisation" of MSO set queries on trees

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A bunch of consequences for *transducers* outputting strings or trees

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Asymptotic study of the growth rate of $f : (\text{strings or trees}) \to \mathbb{N}$

$$\text{growth}[f] : n \in \mathbb{N} \mapsto \max\{f(t) \mid |t| \leqslant n\}$$

Dichotomy phenomena:
- polynomial $\Theta(n^k)$, degree $k \in \mathbb{N}$
- exponential $2^{\Theta(n)}$, degree $+\infty$ by convention

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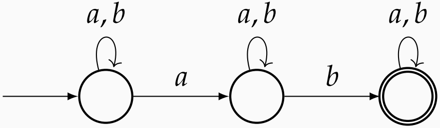
Ambiguity of nondeterministic finite automata

Ambiguity = number of runs of NFA on given input word; for example:



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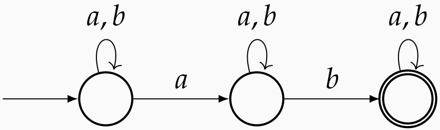

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Theorem (Weber & Seidl 1986)

deg(growth[*ambiguity of* $\mathscr{A}$]) *is well-defined in* $\mathbb{N} \cup \{\infty\}$ (*poly/exp dichotomy*)

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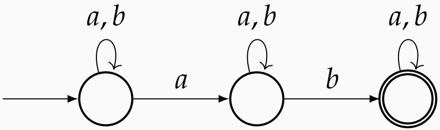

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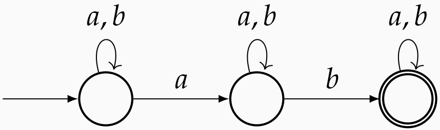

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These bounds are optimal: cf. Karolina Drabik's talk this afternoon

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Theorem (Paul 2015)

$\deg(\text{growth}[$*ambiguity of any tree automaton*$])$ *is well-defined in* $\mathbb{N} \cup \{\infty\}$

Theorem (new)

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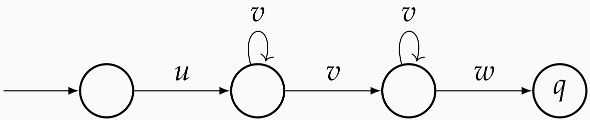
Proof similar to case of strings (next slide):

- branching plays limited role
- words $\rightsquigarrow$ one-hole contexts — e.g. $a(b(\Box), c)$ — in *pumping patterns*

Ambiguity of nondeterministic finite automata: polynomial case

- Define *degrees of states* via pumping patterns, for example:



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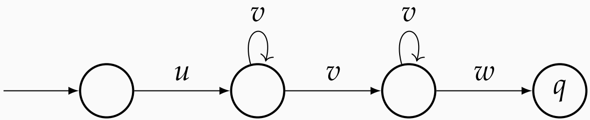

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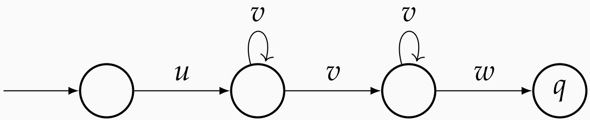

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Lemma (consequence of existing results on finite ambiguity)

There are $O(1)$ runs over a given word/tree with a given set of critical positions.

From automata to Monadic Second-Order logic

nondeterminism $\leftrightarrow$ choice of params X_i in MSO formula $\varphi(X_1, \ldots, X_m)$

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finite-to-one due to previous lemma: $|f_t^{-1}(\{\text{some list}\})| = O(1)$

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- $k = \deg(\text{growth}[\textit{nb of results of query } \varphi]) \in \mathbb{N} \cup \{\infty\}$ *well-def & computable*
- *If* $k < \infty$ *then* $\exists$ "reparameterisation" $f_t \colon \{\textit{results of } \varphi \textit{ on } t\} \to \{\textit{nodes in } t\}^k$ *which is finite-to-one and* MSO-definable by some $\psi(\underbrace{X_1, \dots, X_m}_{\textit{2nd-order}}, \underbrace{z_1, \dots, z_k}_{\textit{1st-order}})$

Proof: f_t(query result) = list of critical positions of corresponding run over t

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Conclusion

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