Slightly non-linear higher-order tree transducers

Lê Thành Dũng (Tito) Nguyễn (Aix-Marseille Univ.)
joint work with Gabriele Vanoni (IRIF, Paris)

STACS 2025 (Marseille $\xrightarrow{\text{online}}$ Jena)

- (Slightly non-)**linear**: as in *linear logic*
- **higher-order**: as in functional programming / λ-*calculus*

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Comparing the expressive power of automata-like devices:

storing λ-terms vs. more conventional

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Comparing the expressive power of automata-like devices:

storing λ-terms vs. more conventional

First: conventional examples on strings

Two-way automata

Transitions: update finite state + move left/right depending on new state

Example: states $Q = \{q_1^{\rightarrow}, q_2^{\leftarrow}, q_3^{\leftarrow}\}$, initial state $q_1^{\rightarrow}$

$$q_1^{\rightarrow}, (a|b) \mapsto q_1^{\rightarrow} \qquad q_1^{\rightarrow}, c \mapsto q_2^{\leftarrow} \qquad q_2^{\leftarrow}, (a|b|c) \mapsto q_3^{\leftarrow} \qquad q_3^{\leftarrow}, b \mapsto \text{accept}$$

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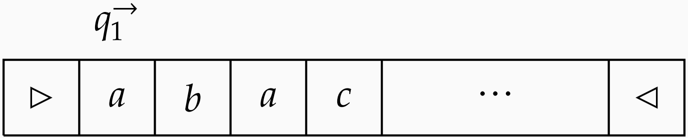

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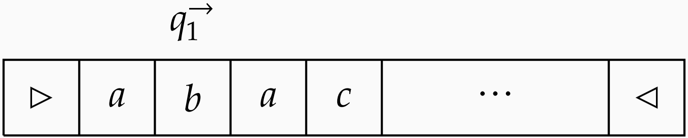

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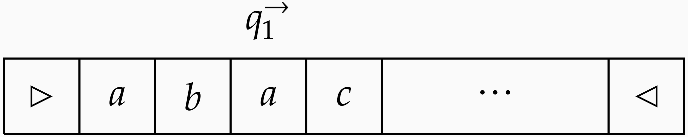

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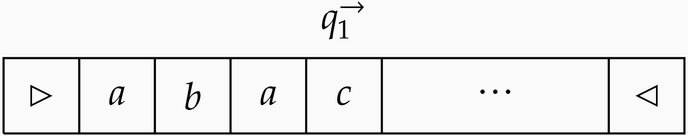

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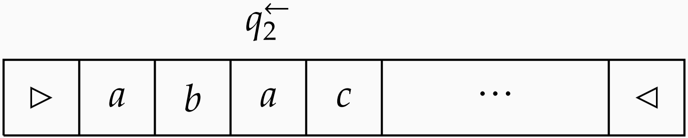

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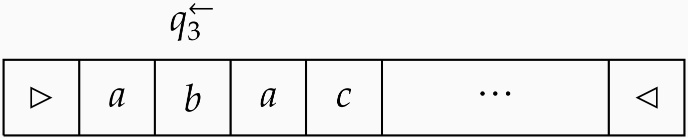

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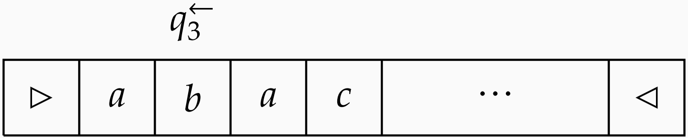

Theorem (Rabin & Scott / Shepherdson 1959)

Two-way automata ≡ one-way automata (≡ regular languages)

⟶ rightfully belong to “finite-state computation”

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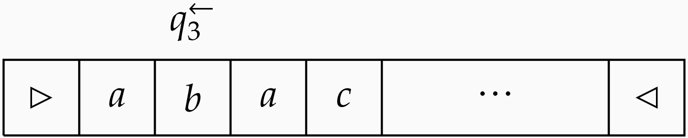

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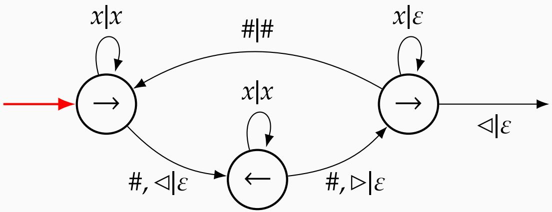
Two-way automata ≡ one-way automata (≡ regular languages)

⟶ rightfully belong to "finite-state computation"

⇒ so does their extension with string outputs

Two-way transducers

Example: $w_1\# \ldots \# w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \ldots \# w_n \cdot \texttt{reverse}(w_n)$



$(x \in \{a, b, c\})$

$\triangleright$	a	b	c	#	b	a	c	#	c	b	$\triangleleft$

Output:

Two-way transducers

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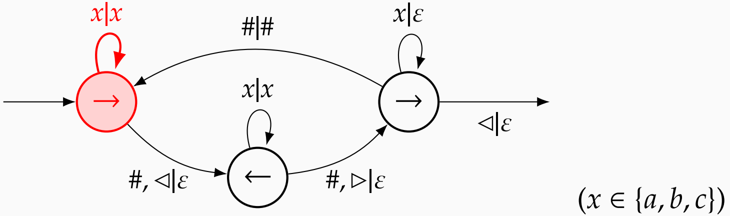
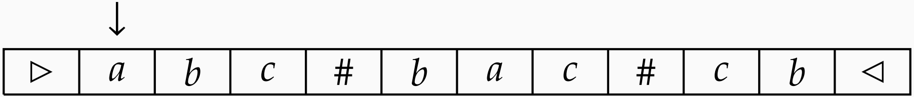


$(x \in \{a, b, c\})$



Output:

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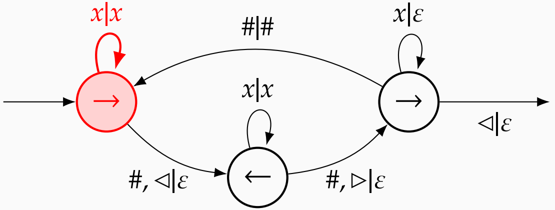
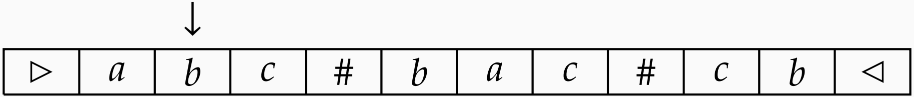


$(x \in \{a, b, c\})$



Output: a

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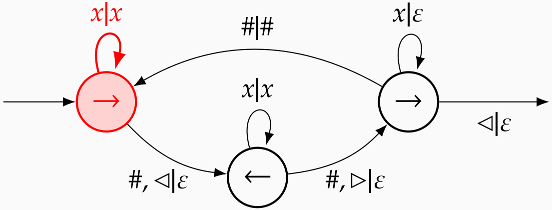

$(x \in \{a, b, c\})$

			↓								
$\rhd$	a	b	c	#	b	a	c	#	c	b	$\lhd$

Output: ab

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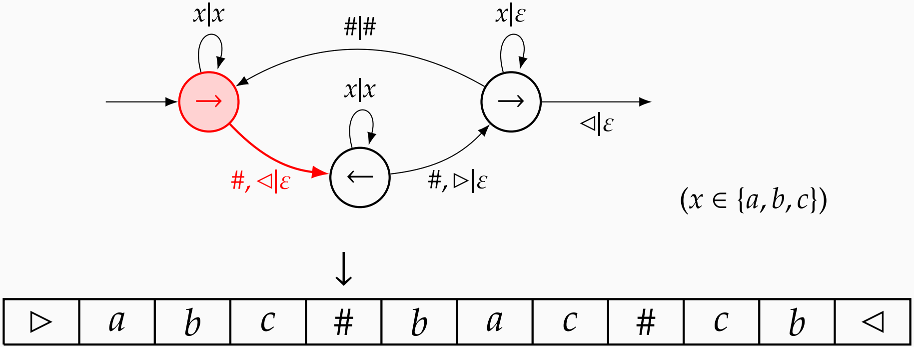

Output: abc

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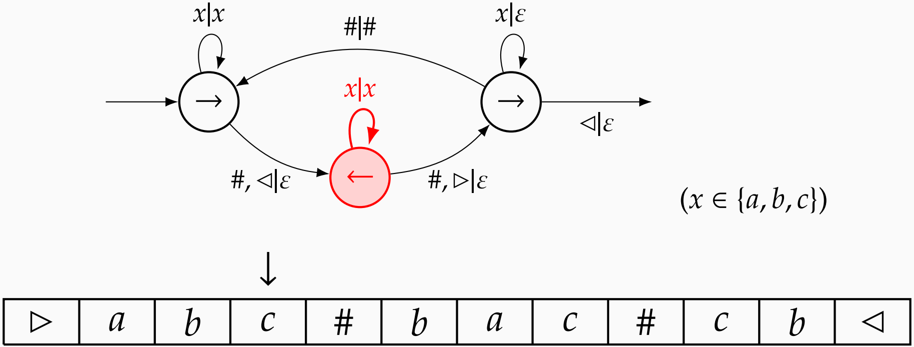

Output: *abc*

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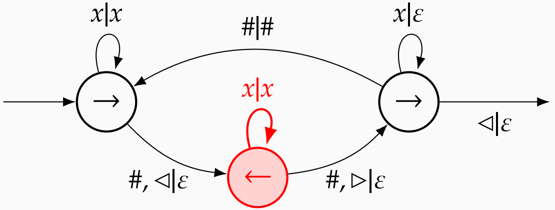
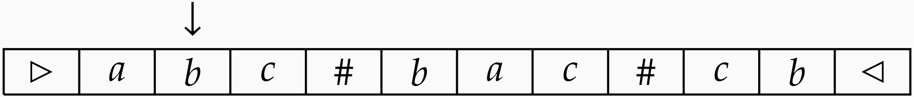


$(x \in \{a, b, c\})$



$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $abcc$

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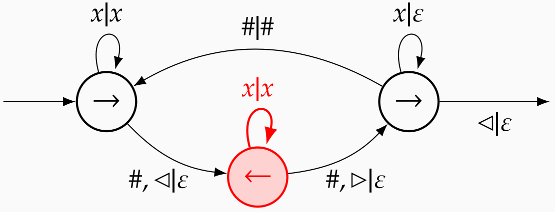

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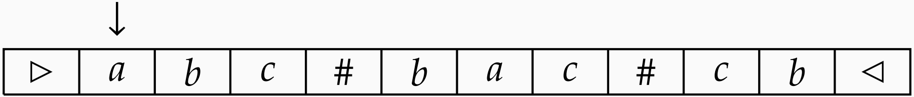

Output: *abccb*

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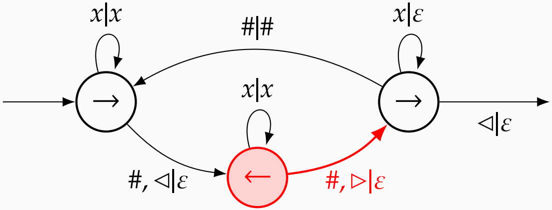

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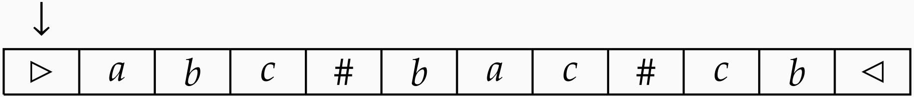

Output: *abccba*

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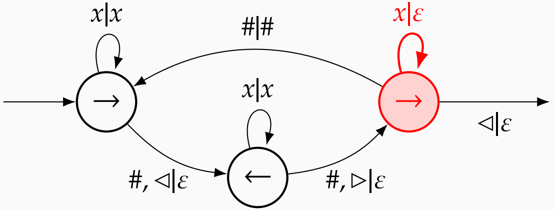

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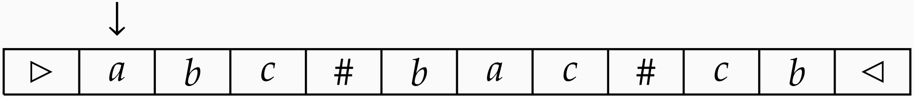

Output: *abccba*

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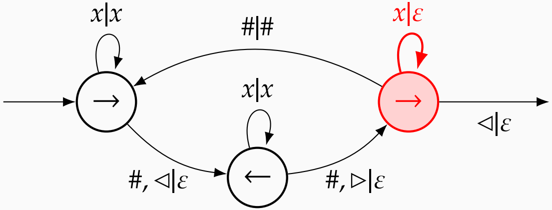

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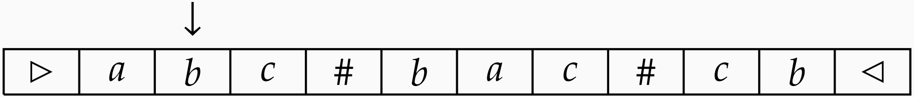

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Output: *abccba*

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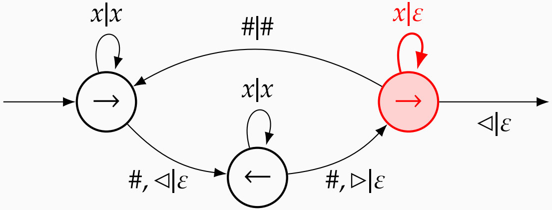

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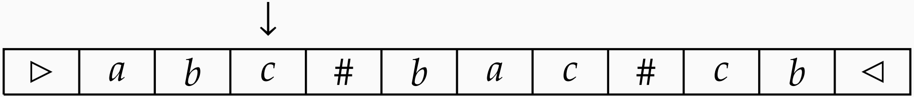

Output: $abccba$

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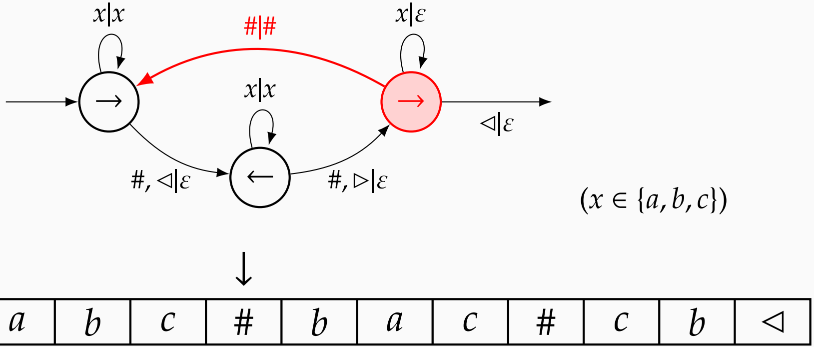

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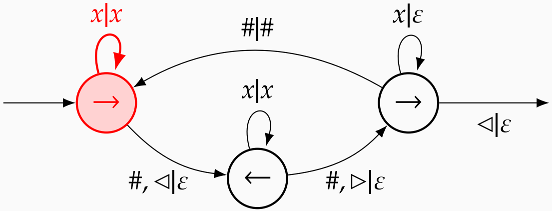

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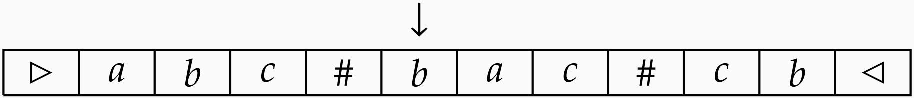

Output: *abccba#*

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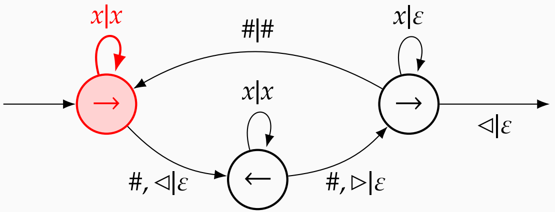

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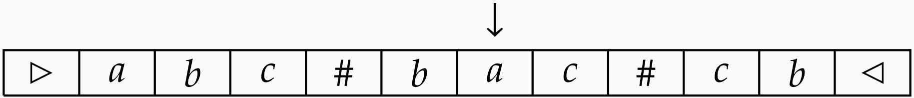

Output: $abccba\#b$

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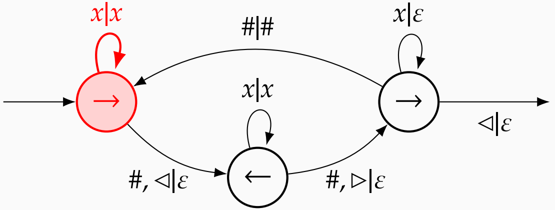

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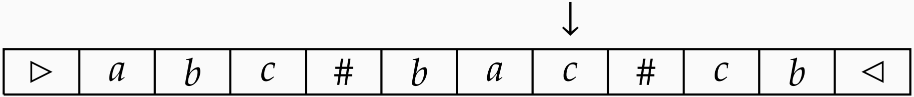

$\triangleright$	a	b	c	#	b	a	c	#	c	b	$\triangleleft$

Output: *abccba#ba*

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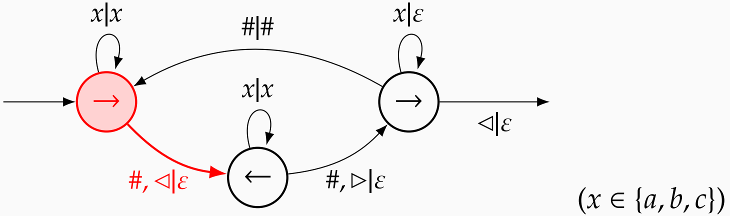

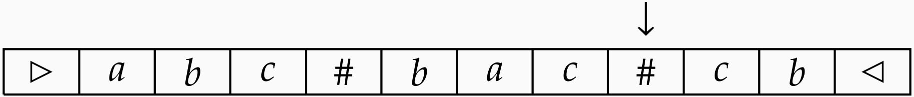

Output: *abccba#bac*

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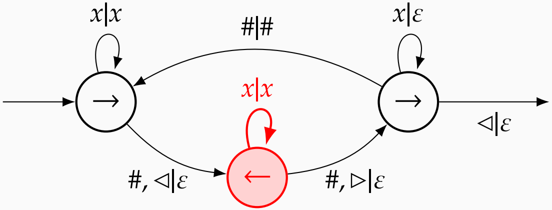

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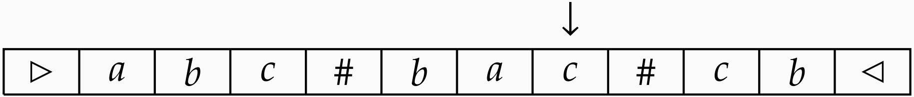

$\rhd$	a	b	c	#	b	a	c	#	c	b	$\lhd$

Output: *abccba#bac*

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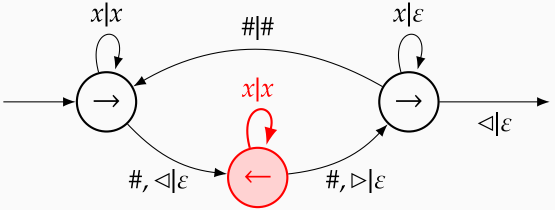

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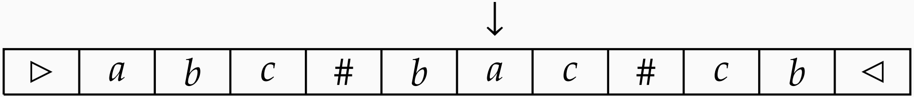

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Output: $abccba\#bacc$

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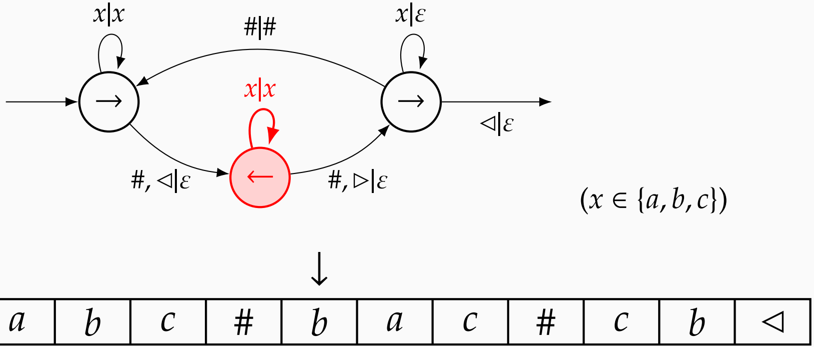

$(x \in \{a, b, c\})$

Output: *abccba#bacca*

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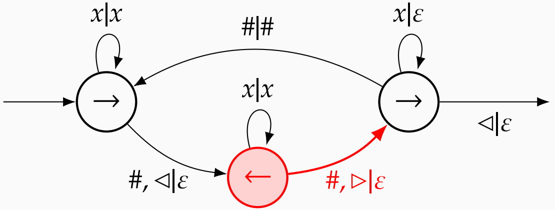
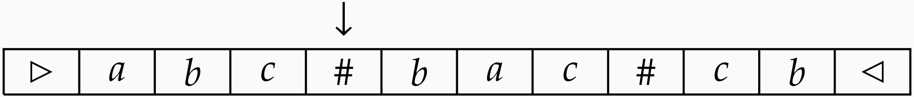


$(x \in \{a, b, c\})$

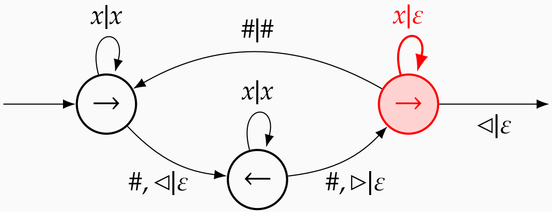


$\rhd$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\lhd$

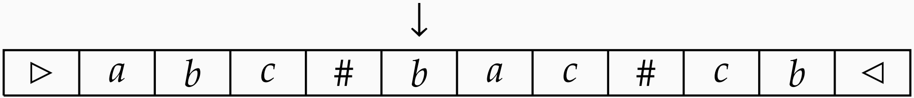
Output: $abccba\#baccab$

Two-way transducers

Example: $w_1\# \dots \#w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \dots \#w_n \cdot \texttt{reverse}(w_n)$



$(x \in \{a, b, c\})$



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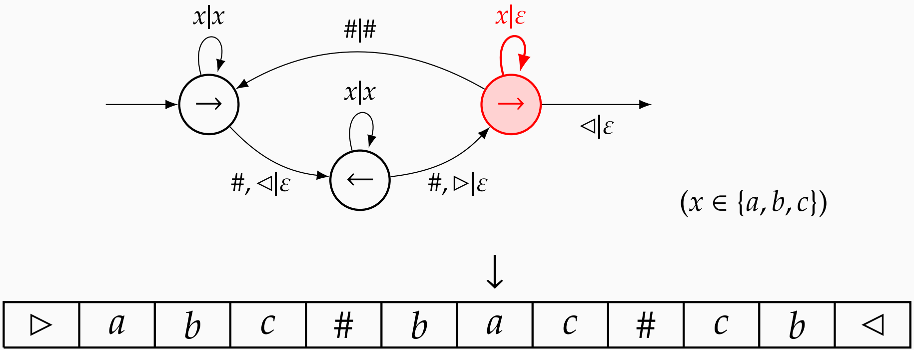

Output: *abccba#baccab*

Two-way transducers

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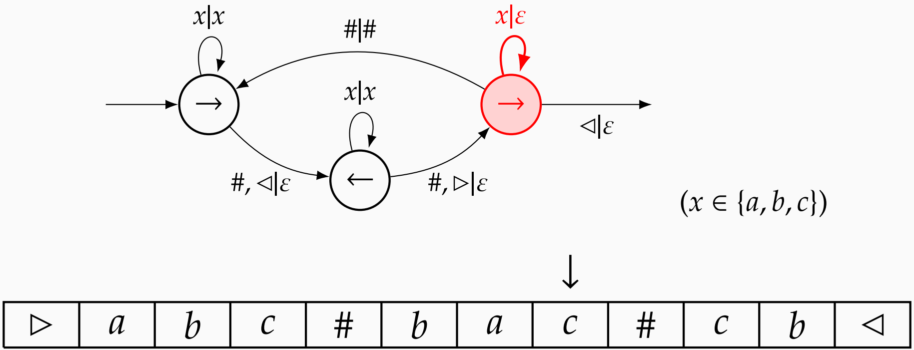

Output: *abccba#baccab*

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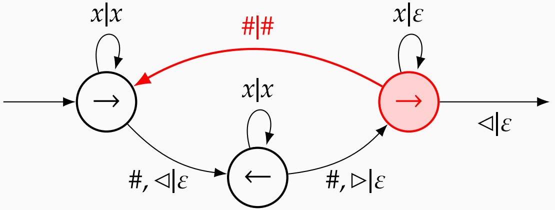
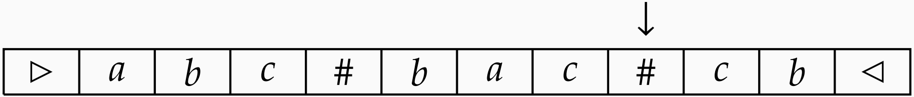


$(x \in \{a, b, c\})$



$\rhd$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\lhd$

Output: *abccba#baccab*

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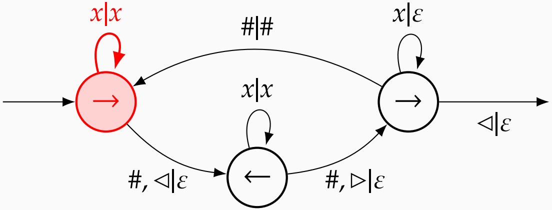
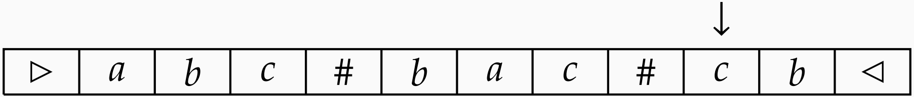


$(x \in \{a, b, c\})$

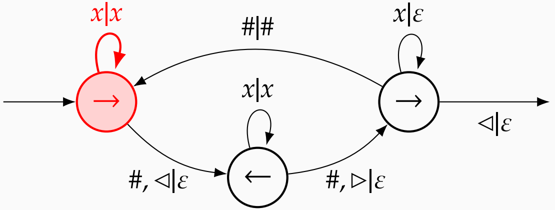


$\triangleright$	a	b	c	#	b	a	c	#	c	b	$\triangleleft$

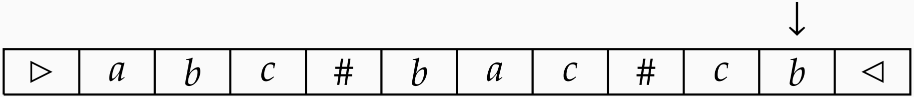
Output: *abccba#baccab#*

Two-way transducers

Example: $w_1\# \dots \# w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \dots \# w_n \cdot \texttt{reverse}(w_n)$



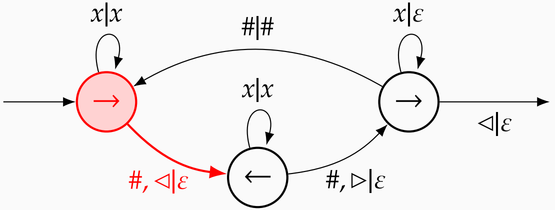
$(x \in \{a, b, c\})$



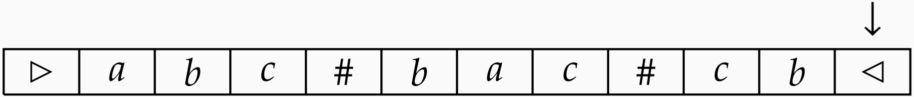
Output: $abccba\#baccab\#c$

Two-way transducers

Example: $w_1\# \dots \# w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \dots \# w_n \cdot \texttt{reverse}(w_n)$



$(x \in \{a, b, c\})$



Output: $abccba\#baccab\#cb$

Two-way transducers

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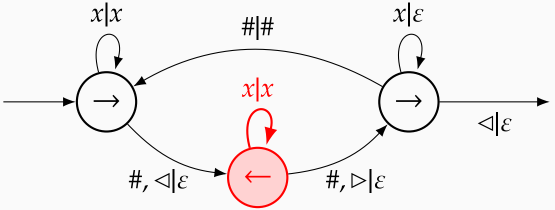
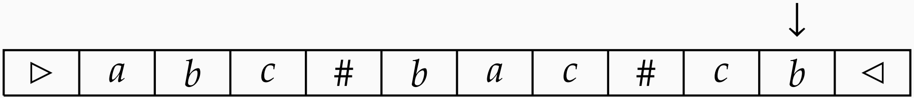


$(x \in \{a, b, c\})$



▷	a	b	c	#	b	a	c	#	c	b	◁

Output: $abccba\#baccab\#cb$

Two-way transducers

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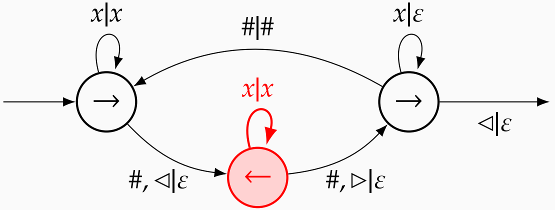
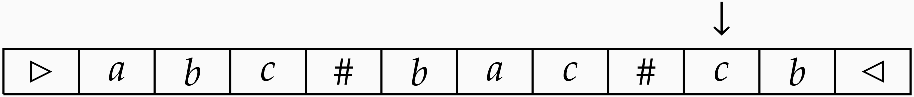


$(x \in \{a, b, c\})$



Output: *abccba#baccab#cbb*

Two-way transducers

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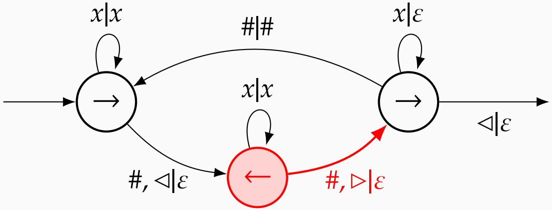

$(x \in \{a, b, c\})$

								↓			
$\triangleright$	a	b	c	#	b	a	c	#	c	b	$\triangleleft$

Output: *abccba#baccab#cbbc*

Two-way transducers

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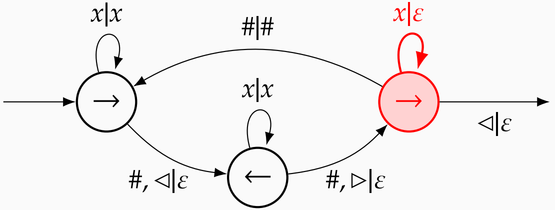
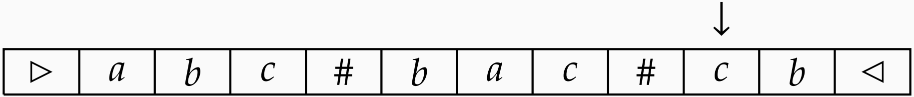


$(x \in \{a, b, c\})$



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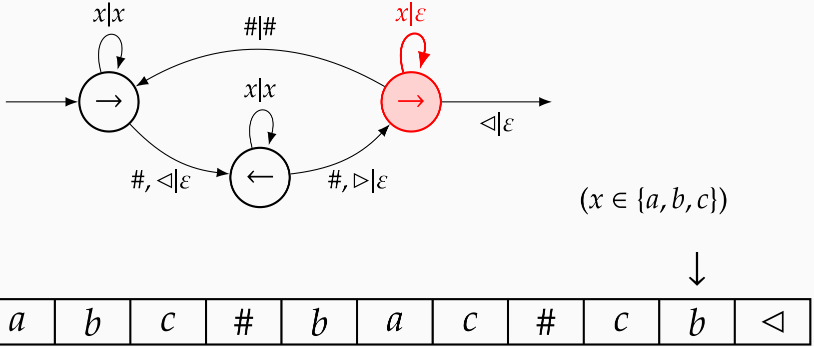

$(x \in \{a, b, c\})$

$\rhd$	a	b	c	#	b	a	c	#	c	b	$\lhd$

Output: *abccba#baccab#cbbc*

Two-way transducers

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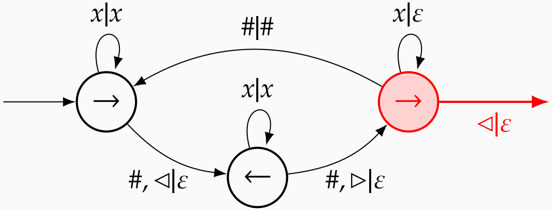
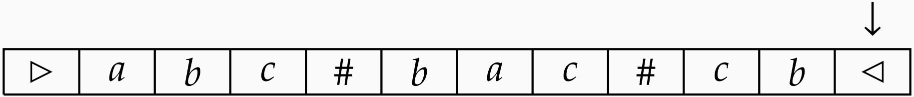


$(x \in \{a, b, c\})$

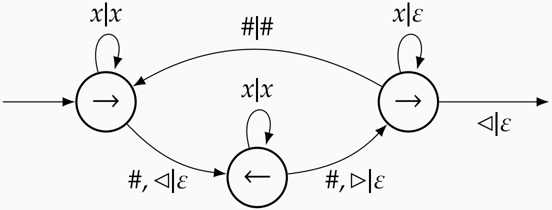


$\rhd$	a	b	c	#	b	a	c	#	c	b	$\lhd$

Output: *abccba#baccab#cbbc*

Two-way transducers are more powerful than one-way

Example: $w_1\# \ldots \# w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \ldots \# w_n \cdot \texttt{reverse}(w_n)$



$(x \in \{a, b, c\})$

$\triangleright$	a	b	c	#	b	a	c	#	c	b	$\triangleleft$

Output: *abccba#baccab#cbbc*

How to realize this with a one-way device?

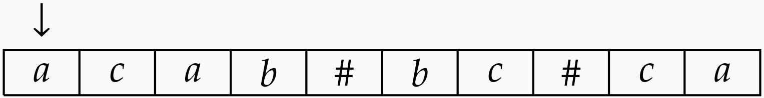
$$\begin{aligned}\texttt{mapReverse}: \quad \{a,b,c,\#\}^* &\quad\longrightarrow\quad \{a,b,c,\#\}^* \\ w_1\# \ldots \# w_n &\quad\longmapsto\quad \texttt{reverse}(w_1)\# \ldots \#\texttt{reverse}(w_n)\end{aligned}$$

a	c	a	b	$\#$	b	c	$\#$	c	a

$$X = \varepsilon \qquad Y = \varepsilon$$

How to realize this with a one-way device?

$$\begin{array}{rrcl} \texttt{mapReverse}: & \{a,b,c,\#\}^* & \to & \{a,b,c,\#\}^* \\ & w_1\#\ldots\#w_n & \mapsto & \texttt{reverse}(w_1)\#\ldots\#\texttt{reverse}(w_n) \end{array}$$

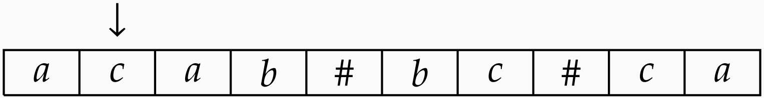


a	c	a	b	#	b	c	#	c	a

$$X = a \qquad Y = \varepsilon$$

How to realize this with a one-way device?

$$\begin{aligned} \texttt{mapReverse}: \quad \{a,b,c,\#\}^* &\quad\to\quad \{a,b,c,\#\}^* \\ w_1\# \dots \# w_n &\quad\mapsto\quad \texttt{reverse}(w_1)\# \dots \#\texttt{reverse}(w_n) \end{aligned}$$

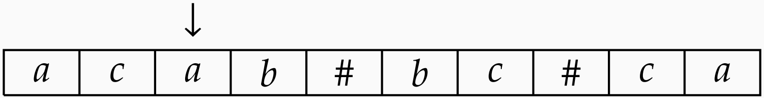


a	c	a	b	#	b	c	#	c	a

$X = ca \qquad Y = \varepsilon$

How to realize this with a one-way device?

$$\begin{aligned} \texttt{mapReverse}: \{a,b,c,\#\}^* &\rightarrow \{a,b,c,\#\}^* \\ w_1\# \dots \# w_n &\mapsto \texttt{reverse}(w_1)\# \dots \#\texttt{reverse}(w_n) \end{aligned}$$

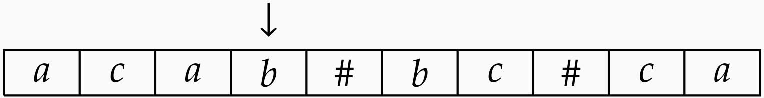


a	c	a	b	#	b	c	#	c	a

$X = aca \qquad Y = \varepsilon$

How to realize this with a one-way device?

$$\begin{array}{rrcl} \texttt{mapReverse}: & \{a,b,c,\#\}^* & \to & \{a,b,c,\#\}^* \\ & w_1\#\ldots\#w_n & \mapsto & \texttt{reverse}(w_1)\#\ldots\#\texttt{reverse}(w_n) \end{array}$$

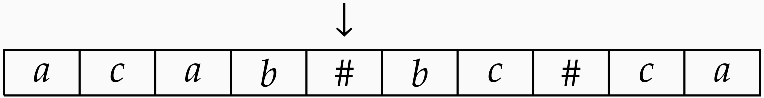


a	c	a	b	$\#$	b	c	$\#$	c	a

$X = baca \qquad Y = \varepsilon$

How to realize this with a one-way device?

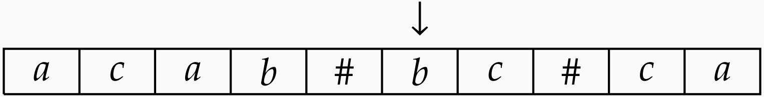
$$\begin{aligned} \texttt{mapReverse}: \{a,b,c,\#\}^* &\rightarrow \{a,b,c,\#\}^* \\ w_1\# \dots \# w_n &\mapsto \texttt{reverse}(w_1)\# \dots \#\texttt{reverse}(w_n) \end{aligned}$$



$X = \varepsilon \qquad Y = baca\#$

How to realize this with a one-way device?

$$\begin{aligned} \texttt{mapReverse} : \quad & \{a,b,c,\#\}^* && \to && \{a,b,c,\#\}^* \\ & w_1\# \dots \# w_n && \mapsto && \texttt{reverse}(w_1)\# \dots \#\texttt{reverse}(w_n) \end{aligned}$$

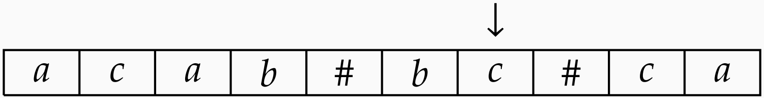


a	c	a	b	$\#$	b	c	$\#$	c	a

$X = b \qquad Y = baca\#$

How to realize this with a one-way device?

$$\begin{aligned}\texttt{mapReverse}: \quad \{a,b,c,\#\}^* &\quad\longrightarrow\quad \{a,b,c,\#\}^* \\ w_1\#\dots\#w_n &\quad\longmapsto\quad \texttt{reverse}(w_1)\#\dots\#\texttt{reverse}(w_n)\end{aligned}$$

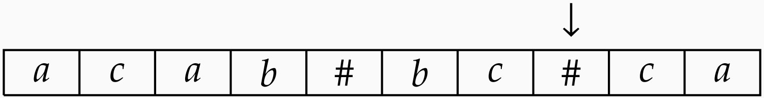


a	c	a	b	#	b	c	#	c	a

$X = cb \qquad Y = baca\#$

How to realize this with a one-way device?

$$\begin{aligned} \texttt{mapReverse}: \quad \{a, b, c, \#\}^* &\quad\to\quad \{a, b, c, \#\}^* \\ w_1\# \dots \# w_n &\quad\mapsto\quad \texttt{reverse}(w_1)\# \dots \#\texttt{reverse}(w_n) \end{aligned}$$

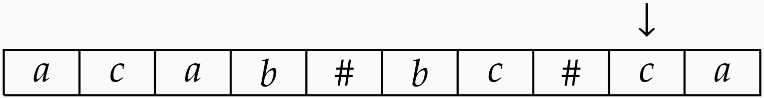


							↓		
a	c	a	b	#	b	c	#	c	a

$X = \varepsilon \qquad Y = baca\#cb\#$

How to realize this with a one-way device?

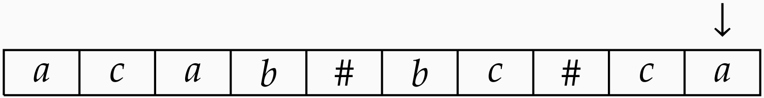
$$\begin{aligned} \texttt{mapReverse}: \quad \{a,b,c,\#\}^* &\quad\to\quad \{a,b,c,\#\}^* \\ w_1\#\dots\#w_n &\quad\mapsto\quad \texttt{reverse}(w_1)\#\dots\#\texttt{reverse}(w_n) \end{aligned}$$



$X = c \qquad Y = baca\#cb\#$

How to realize this with a one-way device?

$$\begin{aligned}\texttt{mapReverse} : \quad \{a,b,c,\#\}^* &\quad\to\quad \{a,b,c,\#\}^* \\ w_1\# \ldots \# w_n &\quad\mapsto\quad \texttt{reverse}(w_1)\# \ldots \#\texttt{reverse}(w_n)\end{aligned}$$



a	c	a	b	#	b	c	#	c	a

$X = ac \qquad Y = baca\#cb\#$

How to realize this with a one-way device?

$$\begin{aligned} \texttt{mapReverse} : \quad \{a,b,c,\#\}^* &\quad\to\quad \{a,b,c,\#\}^* \\ w_1\# \ldots \# w_n &\quad\mapsto\quad \texttt{reverse}(w_1)\# \ldots \#\texttt{reverse}(w_n) \end{aligned}$$

a	c	a	b	$\#$	b	c	$\#$	c	a

$X = ac \qquad Y = baca\#cb\# \qquad \texttt{mapReverse}(\ldots) = YX = baca\#cb\#ac$

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$$\begin{array}{rccl} \texttt{mapReverse}: & \{a,b,c,\#\}^* & \rightarrow & \{a,b,c,\#\}^* \\ & w_1\#\ldots\#w_n & \mapsto & \texttt{reverse}(w_1)\#\ldots\#\texttt{reverse}(w_n) \end{array}$$

a	c	a	b	$\#$	b	c	$\#$	c	a

$$X = ac \qquad Y = baca\#cb\# \qquad \texttt{mapReverse}(\ldots) = YX = baca\#cb\#ac$$

Important point

X, Y concatenable, but **not inspectable** (“if $X[k] = a$ then…”)

⟶ control flow stays finite-state

How to realize this with a one-way device?

$$\begin{array}{rlcl} \texttt{mapReverse}: & \{a,b,c,\#\}^* & \to & \{a,b,c,\#\}^* \\ & w_1\#\ldots\#w_n & \mapsto & \texttt{reverse}(w_1)\#\ldots\#\texttt{reverse}(w_n) \end{array}$$

a	c	a	b	$\#$	b	c	$\#$	c	a

$X = ac$ $\qquad$ $Y = baca\#cb\#$ $\qquad$ $\texttt{mapReverse}(\ldots) = YX = baca\#cb\#ac$

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$\longrightarrow$ control flow stays finite-state

What can you “reasonably” put in memory?

How to realize this with a one-way device?

$$\begin{aligned} \texttt{mapReverse} : \quad \{a, b, c, \#\}^* &\quad\to\quad \{a, b, c, \#\}^* \\ w_1 \# \dots \# w_n &\quad\mapsto\quad \texttt{reverse}(w_1) \# \dots \# \texttt{reverse}(w_n) \end{aligned}$$

a	c	a	b	$\#$	b	c	$\#$	c	a

$X = ac \qquad Y = baca\#cb\# \qquad \texttt{mapReverse}(\dots) = YX = baca\#cb\#ac$

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$\longrightarrow$ control flow stays finite-state

What can you “reasonably” put in memory? **LAMBDA: THE ULTIMATE**

Higher-order tree automata / transducers: simply typed λ-calculus

Bottom-up tree aut.: $a(b(c), c) \mapsto \texttt{accept?}(\delta_a(\delta_b(\delta_c), \delta_c))$ with $\delta_a \colon Q^2 \to Q, \ldots$

Higher-order tree aut.: $a(b(c), c) \mapsto \texttt{accept?}\ (t_a\ (t_b\ t_c)\ t_c)$ with $t_a : A^2 \Rightarrow A, \ldots$

Q finite set vs. $A, B ::= o \mid A \times B \mid A \Rightarrow B$

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Theorem (≃ Hillebrand & Kanellakis 1996 (Damm 1982?))

Higher-order tree automata recognize precisely regular tree languages.

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Theorem (≃ Hillebrand & Kanellakis 1996 (Damm 1982?))

Higher-order tree automata recognize precisely regular tree languages.

Higher-order *transducers*: make o a base type of non-inspectable trees

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Theorem (≃ Hillebrand & Kanellakis 1996 (Damm 1982?))

Higher-order tree automata recognize precisely regular tree languages.

Higher-order *transducers*: make o a base type of non-inspectable trees

memory type $A = o^k$ ▶ k-state *top-down* (sic!) *tree transducer*

Higher-order tree automata / transducers: simply typed λ-calculus

Bottom-up tree aut.: $a(b(c), c) \mapsto \texttt{accept?}(\delta_a(\delta_b(\delta_c), \delta_c))$ with $\delta_a : Q^2 \to Q, \ldots$

Higher-order tree aut.: $a(b(c), c) \mapsto \texttt{accept?}\ (t_a\ (t_b\ t_c)\ t_c)$ with $t_a : A^2 \Rightarrow A, \ldots$

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$A = (o^{\ell_1} \Rightarrow o) \times \cdots \times (o^{\ell_k} \Rightarrow o)$ ▶ k-state *macro tree transducer*, e.g. previous slide!

[Engelfriet & Vogler 1986], staple of "old-school" transducer theory

Higher-order tree automata / transducers: affine types

Problem (feature?): HO tree transducers can express *a lot* of functions
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Theorem (claimed in my PhD; details: [Pradic & Price, MFPS'24])

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i.e. replace $A \Rightarrow B$ by *affine* $A \multimap B$ which *can only use* A *once to produce* B

$\simeq$ "single use restrictions" in automata theory

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Conjecture (N. & Pradic, ICALP'21)

Affine HO tree automata $\subsetneq$ regular tree languages

Results of the paper (1)

Tree-walking: generalization of two-way automata

1 reading head moving around the tree in any direction

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Affine HO tree automata/transducers $\subseteq$ *reversible tree-walking aut./trans.*

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Theorem (N. & Vanoni, this paper)

Affine HO tree automata/transducers	$\subseteq$	*reversible tree-walking aut./trans.*
Almost affine HO tree automata/transducers	$\subseteq$	*tree-walking aut./trans.*

Inexpressivity conjecture from last slide follows from:

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Tree-walking automata $\subsetneq$ *regular tree languages*

Almost affine [Kanazawa]: the base type o can be duplicated, but not the others

↔ "sharing" in the configuration graph of a tree-walking transducer

Results of the paper (2)

Lookaround = can inspect regular information at each node
= preprocessing by very simple transducers / MSO relabeling

Corollary (new proof of [Kanazawa 2008; Gallot, Lemay & Salvati 2020])

Affine HO tree transducers with lookaround	≡	*MSO transductions*
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Other way to overcome inexpressivity [N. & Pradic]: add &/$\oplus$ types

$A \otimes B$ ("multiplicative") vs. $A \mathbin{\&} B$ ("additive")

(better suited to "implicit automata" POV)

Results of the paper (3)

Exponential modality $!A$ makes A duplicable

$A, B ::= o \mid A \multimap B \mid !A \qquad (A \Rightarrow B = !A \multimap B)$

Affine = !-free

Almost affine = '!' only on o

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Almost !-depth 1: '!' only on almost affine types

Theorem (N. & Vanoni, this paper)

Almost !-depth 1 HO tree trans. w/ lookaround ≡ invisible pebble tree transducers
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Thanks for your attention!

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Supplementary material for seminar talk (+ explanations on blackboard)

Affine Interaction Abstract Machine rules

$$\begin{array}{rcl@{\qquad}rcl}
(\mathrm{C}[\underline{t\,u}], T) & \mapsto & (\mathrm{C}[\underline{t}\,u], \bullet\cdot T) & (\mathrm{C}[\overline{t}\,u], \bullet\cdot T) & \mapsto & (\mathrm{C}[\overline{t\,u}], T) \\
(\mathrm{C}[t\,\overline{u}], T) & \mapsto & (\mathrm{C}[\underline{t}\,u], \circ\cdot T) & (\mathrm{C}[\overline{t}\,u], \circ\cdot T) & \mapsto & (\mathrm{C}[t\,\underline{u}], T) \\
(\mathrm{C}[\lambda x.\,\overline{t}], T) & \mapsto & (\mathrm{C}[\overline{\lambda x.\,t}], \bullet\cdot T) & (\mathrm{C}[\underline{\lambda x.\,t}], \bullet\cdot T) & \mapsto & (\mathrm{C}[\lambda x.\,\underline{t}], T) \\
(\mathrm{C}[\lambda x.\,D[\underline{x}]], T) & \mapsto & (\mathrm{C}[\overline{\lambda x.\,D[x]}], \circ\cdot T) & (\mathrm{C}[\underline{\lambda x.\,D[x]}], \circ\cdot T) & \mapsto & (\mathrm{C}[\lambda x.\,D[\overline{x}]], T) \\
(\mathrm{C}[\underline{c}], \bullet^{\mathrm{rk}(c)}\cdot T) & \mapsto & \multicolumn{4}{l}{c\Big((\mathrm{C}[\overline{c}], \circ\cdot T), (\mathrm{C}[\overline{c}], \bullet\cdot\circ\cdot T), \dots, (\mathrm{C}[\overline{c}], \bullet^{\mathrm{rk}(c)-1}\cdot\circ\cdot T)\Big)}
\end{array}$$

Example: $t_a = \lambda\ell.\,\lambda r.\,\lambda x.\,\ell\ (r\ x)$, $\quad t_b = \lambda f.\,\lambda x.\,S\ (f\ x)$, $\quad t_c = S$, $\quad u = \lambda f.\,f\ 0$

$$\begin{aligned}
(\underline{v}, \varepsilon) &\rightsquigarrow (\underline{u}\ (t_a\ (t_b\ t_c)\ t_c), \bullet) \rightsquigarrow ((\lambda f.\ \underline{f\ 0})\ (t_a\ (t_b\ t_c)\ t_c), \varepsilon) \rightsquigarrow ((\lambda f.\ \underline{f}\ 0)\ (\dots), \bullet) \\
&\rightsquigarrow (\overline{(\lambda f.\ f\ 0)}\ (t_a\ (t_b\ t_c)\ t_c), \circ\bullet) \rightsquigarrow (u\ \underline{(t_a\ (t_b\ t_c)\ t_c)}, \bullet) \rightsquigarrow^2 (u\ (\underline{t_a}\ (t_b\ t_c)\ t_c), \bullet\bullet\bullet)
\end{aligned}$$

Additional rules: almost affine case

$$(C[\underline{\texttt{let}\ !x = u\ \texttt{in}\ t}], T) \mapsto (C[\texttt{let}\ !x = u\ \texttt{in}\ \underline{t}], T) \qquad (C[\underline{!t}], T) \mapsto (C[!\underline{t}], T)$$

$$(C[\texttt{let}\ !x = u\ \texttt{in}\ \overline{t}], T) \mapsto (C[\overline{\texttt{let}\ !x = u\ \texttt{in}\ t}], T)$$

$$(C[\texttt{let}\ !x = u\ \texttt{in}\ D[\underline{x}]], T) \mapsto (C[\texttt{let}\ !x = \underline{u}\ \texttt{in}\ D[x]], T)$$

Important: last rule breaks duality/reversibility

Almost !-depth 1 case: add *log* ($\simeq$ *boxes stack*) to IAM configuration

+ single-stack simulation

≈ invisible pebble tree transducer