Computing the polynomial degree of size-to-height increase for macro tree transducers

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joint work with Paul Gallot (Bremen) & Nathan Lhote (Aix–Marseille)

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before that, a special case: *top-down tree transducers*

Top-down tree transducers (DTOP)

Example ("conditional swap"[1]): $f(a(t,u)) = a(f(u), f(t))$, otherwise $f(t) = t$

[1] Inspired from [Alur & D'Antoni 2012]

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The bottom-up view (≈ recursion vs "dynamic programming")

Idea: process a subtree t bottom-up
→ get "register values" $X_0 = q_0\langle t\rangle$, ...

b : $X_0 = X_1 = b(a(b(c),c))$

$a(b(c),c)$: $X_0 = a(c, b(c))$, $X_1 = a(b(c),c)$

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Macro tree transducers (MTTs)

Traditionally: top-down tree transducers with *parameters*, e.g.

$$q_0\langle a(t,u)\rangle \to q_1\langle t\rangle(q_0\langle u\rangle) \qquad q_1\langle a(t,u)\rangle(x) \to a(b(x), \dots)$$

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Bottom-up view: registers store *tree contexts* = λ-terms taking trees as arguments
= trees with "holes"

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Remark: *concatenable strings* are tree contexts ⟶ MTTs (1980s) are the "right" generalization of *streaming string transducers* [Alur & Černy 2010] to trees

$$ac \cdot ab \rightsquigarrow (\lambda x.\, a(c(x))) \circ (\lambda x.\, a(b(x)))$$

Macro tree transducers: examples and growth

In general: $\text{height}(f(t)) = 2^{O(|t|)}$ – up to exponential size-to-height increase!

Using $q_1\langle 0\rangle(x) \to \overbrace{a(x,x)}^{\text{non-linear tree context}}$ and $q_1\langle S(t)\rangle(x) \to \overbrace{q_1\langle t\rangle(q_1\langle t\rangle(x))}^{\text{non-linear use of } q_1\langle t\rangle}$, one can compute:

$$S^n(0) \mapsto \text{complete binary tree of height } 2^n$$

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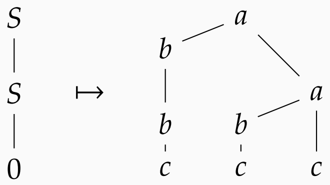
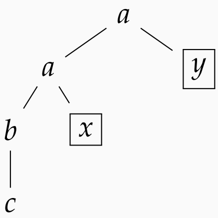
An example with linear (size-to-)height increase + $O(n^2)$ size increase:



Abstracting the output height of an MTT

Tree context $\mapsto$ for each variable x, *highest depth* of x-labeled nodes $\in \mathbb{N} \cup \{-\infty\}$



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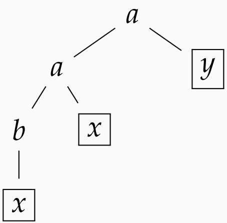

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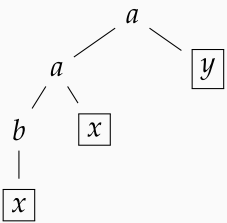

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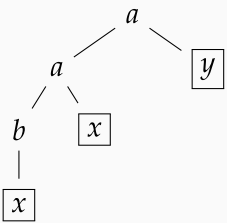

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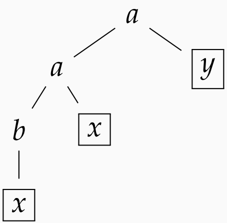

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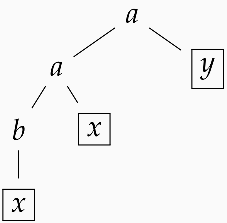

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Use tropical algebra? No, we'll eliminate max

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$\longrightarrow$ nondet. $(\mathbb{N}, +)$-register tree automaton $\mathscr{A}'\colon \mathrm{Tree}(\Sigma) \to \mathscr{P}(\mathbb{N})$, such that

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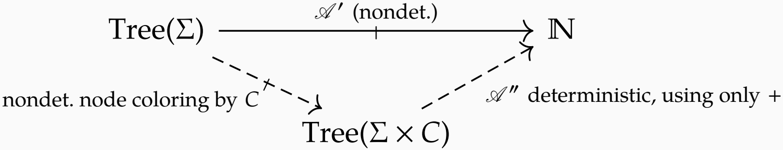

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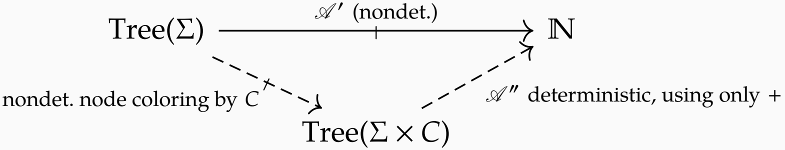

Fact: $\mathscr{A}(t) = O(|t|^k) \iff \mathscr{A}''(t'') = O(|t''|^k)$ for any fixed k

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Equivalently: compute polynomial *degree of ambiguity* of nondet. tree automata

- For strings, well-known, e.g. [Weber & Seidl 1991]
- For trees: not in the literature… almost done in Erik Paul's master thesis

($(\mathbb{N}, \{+\})$-register tree automata = ambiguity of "top-down tree-walking" automata)

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 Thanks for your attention!

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