A complexity gap between pomset logic and system BV, via perfect matchings in digraphs

Or: proof nets according to Christian Retoré

Nguyễn Lê Thành Dũng (a.k.a. Tito) — *nltd@nguyentito.eu* – ÉNS Lyon
joint work with Lutz Straßburger (Inria Saclay)

Journée d'étude pour les 60 ans de Christian Retoré, 15 janvier 2024

What is this about?

Pomset Logic (PL) and *system BV*: 2 logics over the same formulas

A two-decades-old conjecture

These logics are equivalent, i.e. prove the same formulas.

It was known that $(\mathrm{BV} \vdash A) \implies (\mathrm{PL} \vdash A)$.

Our result: refuting the conjecture

There is some formula A such that $\mathrm{BV} \nvdash A$ *but* $\mathrm{PL} \vdash A$.

What is this about?

Pomset Logic (PL) and *system BV*: 2 logics over the same formulas

A two-decades-old conjecture

These logics are equivalent, i.e. prove the same formulas.

It was known that $(\mathrm{BV} \vdash A) \implies (\mathrm{PL} \vdash A)$.

Our result: refuting the conjecture

There is some formula A such that $\mathrm{BV} \nvdash A$ but $\mathrm{PL} \vdash A$.

- What's the point? Why do these logics exist?
- How did we come to think the conjecture should be false?

What is this about?

Pomset Logic (PL) and *system BV*: 2 logics over the same formulas

A two-decades-old conjecture

These logics are equivalent, i.e. prove the same formulas.

It was known that $(\mathrm{BV} \vdash A) \implies (\mathrm{PL} \vdash A)$.

Our result: refuting the conjecture

There is some formula A such that $\mathrm{BV} \nvdash A$ *but* $\mathrm{PL} \vdash A$.

- What's the point? Why do these logics exist?
- How did we come to think the conjecture should be false?

$$A = ((a \lhd b) \otimes (c \lhd d)) \parr ((e \lhd f) \otimes (g \lhd h)) \parr (a^\perp \lhd h^\perp) \parr (e^\perp \lhd b^\perp) \parr (g^\perp \lhd d^\perp) \parr (c^\perp \lhd f^\perp)$$

The classical sequent calculus LK

An usual proof system for classical logic:

- Identity and cut rules
- Logical rules:

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

- Structural rules: contraction and weakening (below) + exchange

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}$$

The classical sequent calculus LK

An usual proof system for classical logic:

- Identity and cut rules
- Logical rules:

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

- Structural rules: contraction and weakening (below) + exchange

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}$$

Remove contraction and weakening $\rightarrow$ *Multiplicative Linear Logic* (MLL)

Multiplicative Linear Logic (MLL)

$$A, B ::= a \mid a^\perp \mid A \otimes B \mid A \parr B$$

Involutive negation *defined* by De Morgan rules:

$$(a^\perp)^\perp = a \qquad (A \otimes B)^\perp = A^\perp \parr B^\perp \qquad (A \parr B)^\perp = A^\perp \otimes B^\perp$$

Sequent calculus: identity and cut rules + exchange + logical rules below:

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

Multiplicative Linear Logic (MLL)

$$A, B ::= a \mid a^{\perp} \mid A \otimes B \mid A \parr B$$

Involutive negation *defined* by De Morgan rules:

$$(a^{\perp})^{\perp} = a \qquad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$

Sequent calculus: identity and cut rules + exchange + logical rules below:

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

MLL is a constructive logic, with non-degenerate denotational semantics!

(corresponds to $*$-autonomous categories)

Multiplicative Linear Logic (MLL)

$$A, B ::= a \mid a^\perp \mid A \otimes B \mid A \parr B$$

Involutive negation *defined* by De Morgan rules:

$$(a^\perp)^\perp = a \qquad (A \otimes B)^\perp = A^\perp \parr B^\perp \qquad (A \parr B)^\perp = A^\perp \otimes B^\perp$$

Sequent calculus: identity and cut rules + exchange + logical rules below:

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

MLL is a constructive logic, with non-degenerate denotational semantics!

(corresponds to $*$-autonomous categories)

$\longrightarrow$ semantics may suggest extensions to the logic

Extensions to Multiplicative Linear Logic

The denotational semantics of MLL in (hyper)coherence spaces suggest:

- The additional *Mix rule* $\dfrac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$ – morally: $A \otimes B \vdash A \parr B$

Extensions to Multiplicative Linear Logic

The denotational semantics of MLL in (hyper)coherence spaces suggest:

- The additional *Mix rule* $\dfrac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$ – morally: $A \otimes B \vdash A \parr B$
- A new connective such that $A \otimes B \vdash A \triangleleft B \vdash A \parr B$, which is:
  non-commutative: $A \triangleleft B \not\equiv B \triangleleft A$ *self-dual:* $(A \triangleleft B)^{\perp} = A^{\perp} \triangleleft B^{\perp}$ (*not* $B^{\perp} \triangleleft A^{\perp}$)

Extensions to Multiplicative Linear Logic

The denotational semantics of MLL in (hyper)coherence spaces suggest:

- The additional *Mix rule* $\dfrac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$ – morally: $A \otimes B \vdash A \parr B$
- A new connective such that $A \otimes B \vdash A \lhd B \vdash A \parr B$, which is:
  non-commutative: $A \lhd B \not\equiv B \lhd A$ *self-dual:* $(A \lhd B)^\perp = A^\perp \lhd B^\perp$ (*not* $B^\perp \lhd A^\perp$)

What should be the inference rules for $\lhd$?

Extensions to Multiplicative Linear Logic

The denotational semantics of MLL in (hyper)coherence spaces suggest:

- The additional *Mix rule* $\dfrac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$ – morally: $A \otimes B \vdash A \parr B$
- A new connective such that $A \otimes B \vdash A \lhd B \vdash A \parr B$, which is:
  non-commutative: $A \lhd B \not\equiv B \lhd A$ *self-dual:* $(A \lhd B)^\perp = A^\perp \lhd B^\perp$ (*not* $B^\perp \lhd A^\perp$)

What should be the inference rules for $\lhd$? Interesting proof-theoretic phenomenon:
to get a reasonable logic, one needs(?) to step *outside the sequent calculus*!

Extensions to Multiplicative Linear Logic

The denotational semantics of MLL in (hyper)coherence spaces suggest:

- The additional *Mix rule* $\dfrac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$ – morally: $A \otimes B \vdash A \parr B$
- A new connective such that $A \otimes B \vdash A \lhd B \vdash A \parr B$, which is:

  non-commutative: $A \lhd B \not\equiv B \lhd A$ *self-dual:* $(A \lhd B)^\perp = A^\perp \lhd B^\perp$ (*not* $B^\perp \lhd A^\perp$)

What should be the inference rules for $\lhd$? Interesting proof-theoretic phenomenon:
to get a reasonable logic, one needs(?) to step *outside the sequent calculus*!

Two conservative extensions of MLL+Mix with $A, B ::= \cdots \mid A \lhd B$

- Pomset Logic (Christian Retoré, early 1990s) — based on *proof nets*
- System BV (Alessio Guglielmi, late 1990s... also anticipated by Retoré!) — 1st application of *deep inference*

Motivations

Guglielmi 2007, *A System of Interaction and Structure* (emphasis mine):

> **It is still open whether the logic in this paper, called *BV*, is the same as pomset logic. We conjecture that it is actually the same logic**, *but one crucial step is still missing, at the time of this writing, in the equivalence proof. This paper is the first in a planned series of 3 papers dedicated to BV.* [...] *In the 3rd part, some of my colleagues will hopefully show the equivalence of BV and pomset logic*

Motivations

Guglielmi 2007, *A System of Interaction and Structure* (emphasis mine):

> **It is still open whether the logic in this paper, called *BV*, is the same as pomset logic. We conjecture that it is actually the same logic**, *but one crucial step is still missing, at the time of this writing, in the equivalence proof. This paper is the first in a planned series of 3 papers dedicated to BV.* [...] *In the 3rd part, some of my colleagues will hopefully show the equivalence of BV and pomset logic*

As announced earlier, this conjecture is false!

Motivations

Guglielmi 2007, *A System of Interaction and Structure* (emphasis mine):

> **It is still open whether the logic in this paper, called *BV*, is the same as pomset logic. We conjecture that it is actually the same logic**, *but one crucial step is still missing, at the time of this writing, in the equivalence proof. This paper is the first in a planned series of 3 papers dedicated to BV.* [...] *In the 3rd part, some of my colleagues will hopefully show the equivalence of BV and pomset logic,* **this way explaining why it was impossible to find a sequent system for pomset logic.**

As announced earlier, this conjecture is false!

Motivations

Guglielmi 2007, *A System of Interaction and Structure* (emphasis mine):

> **It is still open whether the logic in this paper, called *BV*, is the same as pomset logic. We conjecture that it is actually the same logic**, *but one crucial step is still missing, at the time of this writing, in the equivalence proof. This paper is the first in a planned series of 3 papers dedicated to BV.* [...] *In the 3rd part, some of my colleagues will hopefully show the equivalence of BV and pomset logic,* **this way explaining why it was impossible to find a sequent system for pomset logic.**

As announced earlier, this conjecture is false!

$\exists$ formal arguments showing that “traditional sequent calculi cannot express BV”

[Tiu 2006]

A glance at deep inference

A methodology originally introduced for BV;

many other successes in past 2 decades (e.g. cut-free proofs for modal logics)

Deep inference = unary rules applied to subformulas of arbitrary depth:

$$\text{inference rule } \frac{A}{B} \qquad \rightsquigarrow \qquad \text{instances } \frac{S[A]}{S[B]} \text{ for any context } S$$

e.g. $$\frac{A \parr (B \otimes (C \parr D))}{A \parr ((B \otimes C) \parr D)}$$

A glance at deep inference

A methodology originally introduced for BV;

many other successes in past 2 decades (e.g. cut-free proofs for modal logics)

Deep inference = unary rules applied to subformulas of arbitrary depth:

$$\text{inference rule } \frac{A}{B} \qquad \rightsquigarrow \qquad \text{instances } \frac{S[A]}{S[B]} \text{ for any context } S$$

e.g.

$$\cfrac{\cfrac{(A \parr B) \otimes (C \parr D)}{A \parr (\color{red}{B \otimes (C \parr D)})}}{A \parr (\color{red}{(B \otimes C) \parr D})} \qquad \Longleftrightarrow \qquad \frac{\vdash A, B \qquad \vdash C, D}{\vdash A, B \otimes C, D}$$

A glance at deep inference

A methodology originally introduced for BV;

many other successes in past 2 decades (e.g. cut-free proofs for modal logics)

Deep inference = unary rules applied to subformulas of arbitrary depth:

$$\text{inference rule } \frac{A}{B} \qquad \rightsquigarrow \qquad \text{instances } \frac{S[A]}{S[B]} \text{ for any context } S$$

e.g.

$$\frac{\dfrac{(A \parr B) \otimes (C \parr D)}{A \parr (B \otimes (C \parr D))}}{A \parr ((B \otimes C) \parr D)} \qquad \Longleftrightarrow \qquad \frac{\vdash A, B \qquad \vdash C, D}{\vdash A, B \otimes C, D}$$

(compare with rewriting systems, or functoriality in categorical logic)

Deep inference for MLL+Mix and BV

- Identity rules: $\overline{\mathbf{I}}$ and $\dfrac{\mathbf{I}}{a \parr a^\perp}$
- MLL+Mix: rules for assoc/comm. of $\otimes$, $\parr$ + unitality ($A \otimes \mathbf{I} \equiv A \parr \mathbf{I} \equiv A$) +

$$\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \text{(where } A, B, C \text{ may be equal to } \mathbf{I}\text{)}$$

Deep inference for MLL+Mix and BV

- Identity rules: $\overline{\mathbf{I}}$ and $\dfrac{\mathbf{I}}{a \parr a^{\perp}}$
- MLL+Mix: rules for assoc/comm. of $\otimes, \parr$ + unitality ($A \otimes \mathbf{I} \equiv A \parr \mathbf{I} \equiv A$) +

$$\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \text{(where } A, B, C \text{ may be equal to } \mathbf{I}\text{)}$$

- BV: MLL+Mix rules for associativity/unitality of $\triangleleft$ (*not* commutativity!) +

$$\frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)} \qquad \text{(where } A, B, C, D \text{ may be equal to } \mathbf{I}\text{)}$$

Deep inference for MLL+Mix and BV

- Identity rules: $\overline{\mathbf{I}}$ and $\dfrac{\mathbf{I}}{a \parr a^\perp}$
- MLL+Mix: rules for assoc/comm. of $\otimes, \parr$ + unitality ($A \otimes \mathbf{I} \equiv A \parr \mathbf{I} \equiv A$) +

$$\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \text{(where } A, B, C \text{ may be equal to } \mathbf{I}\text{)}$$

- BV: MLL+Mix rules for associativity/unitality of $\lhd$ (*not* commutativity!) +

$$\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \qquad \text{(where } A, B, C, D \text{ may be equal to } \mathbf{I}\text{)}$$

- one can add a cut $\dfrac{a \otimes a^\perp}{\mathbf{I}}$ and the dual $\otimes/\lhd$ rule: *system SBV*

conservativity of SBV over BV $\approx$ cut-elimination

Deep inference for MLL+Mix and BV

- Identity rules: $\overline{\mathbf{I}}$ and $\dfrac{\mathbf{I}}{a \parr a^\perp}$
- MLL+Mix: rules for assoc/comm. of $\otimes$, $\parr$ + unitality ($A \otimes \mathbf{I} \equiv A \parr \mathbf{I} \equiv A$) +

$$\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \text{(where } A, B, C \text{ may be equal to } \mathbf{I}\text{)}$$

- BV: MLL+Mix rules for associativity/unitality of $\triangleleft$ (*not* commutativity!) +

$$\frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)} \qquad \text{(where } A, B, C, D \text{ may be equal to } \mathbf{I}\text{)}$$

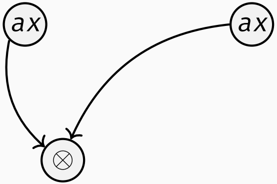
- one can add a cut $\dfrac{a \otimes a^\perp}{\mathbf{I}}$ and the dual $\otimes/\triangleleft$ rule: *system SBV*

  conservativity of SBV over BV $\approx$ cut-elimination
- unit-less SBV rules (except identity/cut) first proposed by Retoré!

  Pomset logic as a calculus of directed cographs, 1999 (Inria Research Report)

Proof nets for Multiplicative Linear Logic

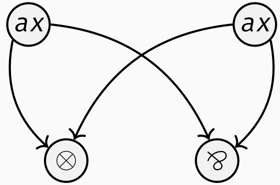
The proof system for Pomset Logic extends the graphical syntax of MLL *proof nets*



$$\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,\mathsf{ax} \qquad \dfrac{}{\vdash B, B^{\perp}}\,\mathsf{ax}}{\vdash A \otimes B, A^{\perp}, B^{\perp}}\,\otimes$$

The proof system for Pomset Logic extends the graphical syntax of MLL *proof nets*

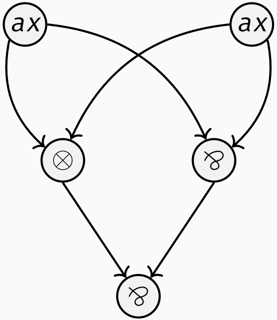
$$\dfrac{\dfrac{}{\vdash A, A^\perp}\,\mathsf{ax} \qquad \dfrac{}{\vdash B, B^\perp}\,\mathsf{ax}}{\dfrac{\vdash A \otimes B, A^\perp, B^\perp}{\vdash A \otimes B, A^\perp \parr B^\perp}\,\parr}\,\otimes$$



Proof nets for Multiplicative Linear Logic

The proof system for Pomset Logic extends the graphical syntax of MLL *proof nets*

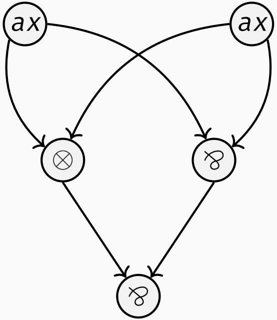
$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,\mathsf{ax} \qquad \dfrac{}{\vdash B, B^{\perp}}\,\mathsf{ax}}{\vdash A \otimes B, A^{\perp}, B^{\perp}}\,\otimes}{\vdash A \otimes B, A^{\perp} \parr B^{\perp}}\,\parr}{\vdash (A \otimes B) \parr (A^{\perp} \parr B^{\perp})}\,\parr$$



Proof nets for Multiplicative Linear Logic

The proof system for Pomset Logic extends the graphical syntax of MLL *proof nets*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,ax \qquad \dfrac{}{\vdash B, B^{\perp}}\,ax}{\vdash A \otimes B, A^{\perp}, B^{\perp}}\,\otimes}{\vdash A \otimes B, A^{\perp} \parr B^{\perp}}\,\parr}{\vdash (A \otimes B) \parr (A^{\perp} \parr B^{\perp})}\,\parr$$



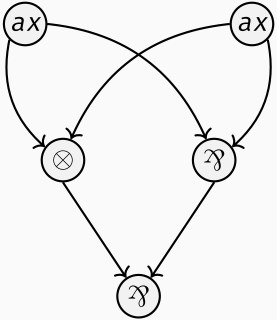
No distinction between $\otimes$ and $\parr$ $\longrightarrow$ not all graphs correspond to correct proofs

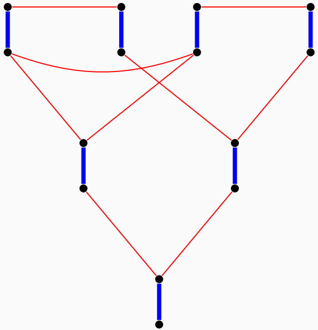
$\longrightarrow$ need a *correctness criterion*

In addition to Pomset Logic, Retoré also invented in the 1990s…

A translation MLL+Mix proof nets → graphs equipped with *perfect matchings*

(for linear logicians: reformulation of Danos–Regnier switching criterion)





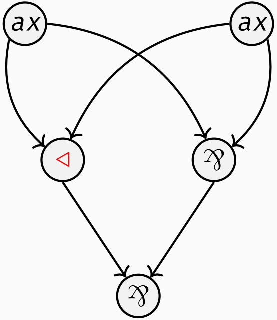
comes from a correct proof $\iff$ no alternating elementary cycle

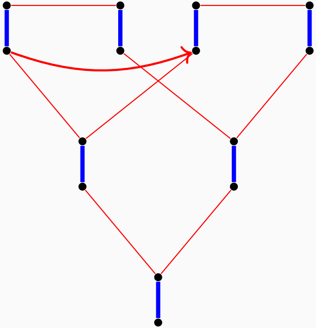
In addition to Pomset Logic, Retoré also invented in the 1990s...

A translation MLL+Mix proof nets $\to$ graphs equipped with *perfect matchings*

(for linear logicians: reformulation of Danos–Regnier switching criterion)

Pomset logic: add *directed* edges to handle non-commutativity





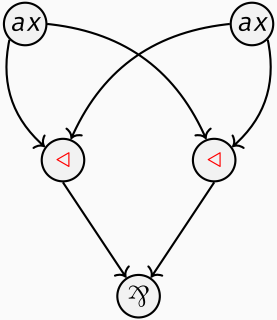
comes from a correct proof $\iff$ no alternating elementary cycle

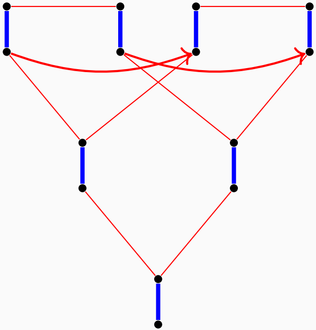
In addition to Pomset Logic, Retoré also invented in the 1990s...

A translation MLL+Mix proof nets → graphs equipped with *perfect matchings*

(for linear logicians: reformulation of Danos–Regnier switching criterion)

Pomset logic: add *directed* edges to handle non-commutativity





comes from a correct proof $\iff$ no alternating elementary cycle

Complexity of proof net correctness

In previous work (*Log. Methods Comput. Sci.* 2020) I brought proof nets and perfect matchings even closer by exhibiting a converse translation ($\leftarrow$)

$\longrightarrow$ we can apply results from mainstream combinatorics to study these logics!

Complexity of proof net correctness

In previous work (*Log. Methods Comput. Sci.* 2020) I brought proof nets and perfect matchings even closer by exhibiting a converse translation ($\leftarrow$)

$\longrightarrow$ we can apply results from mainstream combinatorics to study these logics!

This led us to a surprising realization:

Theorem (N. & Straßburger – LMCS paper, published last month)

Provability in pomset logic is strictly harder *than in BV unless* $\mathsf{NP} = \mathsf{coNP}$.

(*more precisely:* Σ_2^p*-complete vs* NP*-complete*)

- In BV, the length of proofs is polynomially bounded
- It's known that finding constrained cycles in directed graphs is often hard

  (inspiration: Gourvès et al. 2013, *Complexity of trails, paths and circuits in arc-colored digraphs*)

Complexity of proof net correctness

In previous work (*Log. Methods Comput. Sci.* 2020) I brought proof nets and perfect matchings even closer by exhibiting a converse translation ($\leftarrow$)

$\longrightarrow$ we can apply results from mainstream combinatorics to study these logics!

This led us to a surprising realization:

Theorem (N. & Straßburger – LMCS paper, published last month)

Provability in pomset logic is strictly harder *than in BV unless* $\mathsf{NP} = \mathsf{coNP}$.

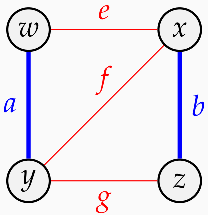
(*more precisely:* Σ_2^{p}*-complete vs* NP*-complete*)

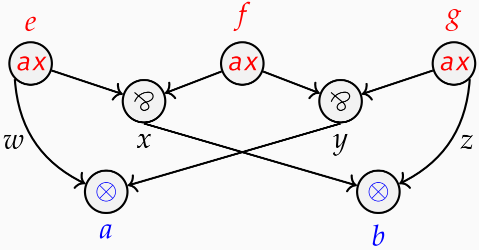
- In BV, the length of proofs is polynomially bounded
- It's known that finding constrained cycles in directed graphs is often hard

  (inspiration: Gourvès et al. 2013, *Complexity of trails, paths and circuits in arc-colored digraphs*)

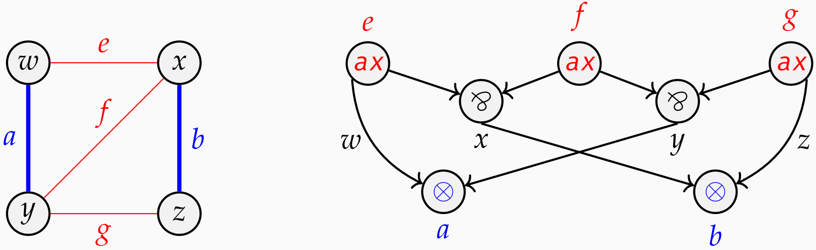
$\longrightarrow$ Suddenly, Guglielmi's conjecture looked less plausible...

Reduction perfect matchings → proof structures (FSCD 2018 / LMCS 2020)





Reduction perfect matchings → proof structures (FSCD 2018 / LMCS 2020)



Extends to perfect matchings in general *directed* graphs → *pomset logic* proof structures, using the non-commutative $\triangleleft$ to build a "directed ax" gadget. Hence:

Lemma

There is a PTIME *reduction:* $\exists$ *directed æ-cycle* $\rightsquigarrow$ *pomset logic proof net* incorrectness.

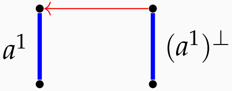
Directed axioms and causality

Aleks Kissinger & Will Simmons (Oxford) consider categories modelling higher-order causal processes (obtained via double-gluing). They show that the "logic of these categories" is precisely a conservative extension of pomset logic.

(*An exact logic for compatibility of higher-order causal structures*, in preparation)

$$A, B ::= a^1 \mid (a^1)^\perp \mid a \mid a^\perp \mid A \otimes B \mid A \parr B \mid A \lhd B$$

Special atoms a^1 represent 1st-order interfaces to processes, with directed axioms:



$\implies$ easier reduction [$\exists$ directed æ-cycle] $\rightsquigarrow$ incorrectness

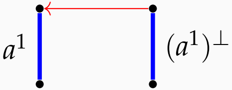
Directed axioms and causality

Aleks Kissinger & Will Simmons (Oxford) consider categories modelling higher-order causal processes (obtained via double-gluing). They show that the "logic of these categories" is precisely a conservative extension of pomset logic.

(*An exact logic for compatibility of higher-order causal structures*, in preparation)

$$A, B ::= a^1 \mid (a^1)^\perp \mid a \mid a^\perp \mid A \otimes B \mid A \parr B \mid A \lhd B$$

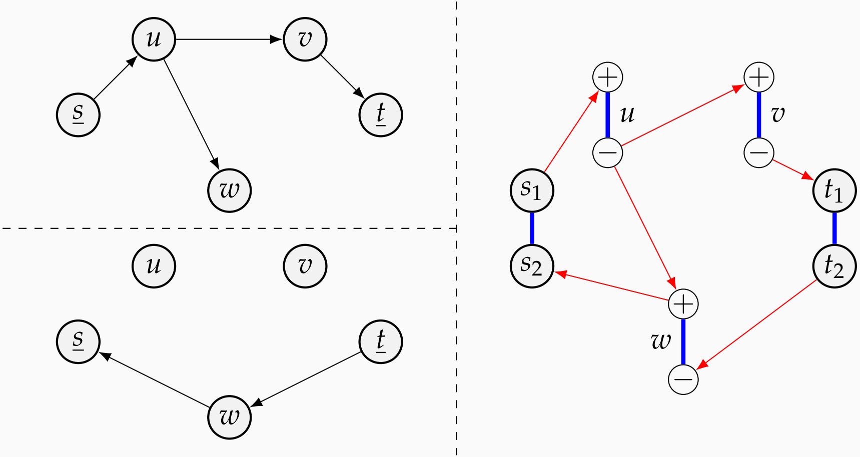
Special atoms a^1 represent 1st-order interfaces to processes, with directed axioms:



$\Longrightarrow$ easier reduction [$\exists$ directed æ-cycle] $\rightsquigarrow$ incorrectness

Next goal: show that finding directed æ-cycles is NP-hard

Reduction from another graph-theoretic problem ("elementary round-trip")



cycle $s \xrightarrow{\text{top}} t \xrightarrow{\text{bottom}} s$ *without repeating vertex* $\leftrightsquigarrow$ æ-cycle

Hardness results

The "elementary round-trip" problem is NP-complete $(\overbrace{\text{by reduction from 3SAT}}^{\text{backup slide}})$, so:

Theorem

Pomset proof net correctness is coNP-complete.

Hardness results

The "elementary round-trip" problem is NP-complete $\overbrace{\text{(by reduction from 3SAT)}}^{\text{backup slide}}$, so:

Theorem

Pomset proof net correctness is coNP*-complete.*

We can reduce a Π_2^{p}-complete variant of elementary round-trip involving the $\underbrace{\text{“switchings” of two “paired graphs”}}_{\textit{à la}\text{ Danos–Regnier}}$ to pomset *non-provability*, therefore:

Theorem

Pomset logic provability is Σ_2^{p}*-complete.*

Remark: here paired graphs / switchings are *not* related to the correctness criterion but to the choice of plugging of axiom links

Conclusion

Retoré's *Pomset Logic* (PL) and Guglielmi's *BV*: 2 logics over the same formulas, from the 1990s, conservatively extending Multiplicative Linear Logic with Mix

Our result [N. & Straßburger]: refuting Guglielmi's two-decades-old conjecture

- *There is some formula A such that* $\mathrm{BV} \nvdash A$ *but* $\mathrm{PL} \vdash A$.

$$A = ((a \lhd b) \otimes (c \lhd d)) \parr ((e \lhd f) \otimes (g \lhd h)) \parr (a^\perp \lhd h^\perp) \parr (e^\perp \lhd b^\perp) \parr (g^\perp \lhd d^\perp) \parr (c^\perp \lhd f^\perp)$$

  Causally meaningful variant (K.–S.): $(((p^1)^\perp \lhd q^1) \otimes ((r^1)^\perp \lhd s^1)) \parr (((q^1)^\perp \lhd r^1) \otimes ((s^1)^\perp \lhd p^1))$

- Moreover, "$\mathrm{BV} \vdash A$?" is NP-complete while "$\mathrm{PL} \vdash A$?" is Σ_2^{p}-complete.

Conclusion

Retoré's *Pomset Logic* (PL) and Guglielmi's *BV*: 2 logics over the same formulas, from the 1990s, conservatively extending Multiplicative Linear Logic with Mix

Our result [N. & Straßburger]: refuting Guglielmi's two-decades-old conjecture

- *There is some formula A such that* $\mathrm{BV} \nvdash A$ *but* $\mathrm{PL} \vdash A$.

$$A = ((a \lhd b) \otimes (c \lhd d)) \parr ((e \lhd f) \otimes (g \lhd h)) \parr (a^\perp \lhd h^\perp) \parr (e^\perp \lhd b^\perp) \parr (g^\perp \lhd d^\perp) \parr (c^\perp \lhd f^\perp)$$

Causally meaningful variant (K.–S.): $(((p^1)^\perp \lhd q^1) \otimes ((r^1)^\perp \lhd s^1)) \parr (((q^1)^\perp \lhd r^1) \otimes ((s^1)^\perp \lhd p^1))$

- Moreover, "$\mathrm{BV} \vdash A$?" is NP-complete while "$\mathrm{PL} \vdash A$?" is Σ_2^{p}-complete.

- These logics exemplify two proof-theoretic paradigms going by necessity beyond the sequent calculus: *proof nets* and *deep inference*.

Conclusion

Retoré's *Pomset Logic* (PL) and Guglielmi's *BV*: 2 logics over the same formulas, from the 1990s, conservatively extending Multiplicative Linear Logic with Mix

Our result [N. & Straßburger]: refuting Guglielmi's two-decades-old conjecture

- *There is some formula A such that* $\mathrm{BV} \nvdash A$ *but* $\mathrm{PL} \vdash A$*.*

  $$A = ((a \triangleleft b) \otimes (c \triangleleft d)) \parr ((e \triangleleft f) \otimes (g \triangleleft h)) \parr (a^\perp \triangleleft h^\perp) \parr (e^\perp \triangleleft b^\perp) \parr (g^\perp \triangleleft d^\perp) \parr (c^\perp \triangleleft f^\perp)$$

  Causally meaningful variant (K.–S.): $(((p^1)^\perp \triangleleft q^1) \otimes ((r^1)^\perp \triangleleft s^1)) \parr (((q^1)^\perp \triangleleft r^1) \otimes ((s^1)^\perp \triangleleft p^1))$
- Moreover, "$\mathrm{BV} \vdash A$?" is NP-complete while "$\mathrm{PL} \vdash A$?" is Σ_2^{p}-complete.

- These logics exemplify two proof-theoretic paradigms going by necessity beyond the sequent calculus: *proof nets* and *deep inference*.
- We realized that the conjecture was very probably false thanks to connections with mainstream graph theory (also initiated by Retoré!).

Conclusion

Joyeux anniversaire Christian !

Retoré's *Pomset Logic* (PL) and Guglielmi's *BV*: 2 logics over the same formulas, from the 1990s, conservatively extending Multiplicative Linear Logic with Mix

Our result [N. & Straßburger]: refuting Guglielmi's two-decades-old conjecture

- *There is some formula A such that* $\mathrm{BV} \nvdash A$ *but* $\mathrm{PL} \vdash A$.

$$A = ((a \lhd b) \otimes (c \lhd d)) \parr ((e \lhd f) \otimes (g \lhd h)) \parr (a^\perp \lhd h^\perp) \parr (e^\perp \lhd b^\perp) \parr (g^\perp \lhd d^\perp) \parr (c^\perp \lhd f^\perp)$$

  Causally meaningful variant (K.–S.): $(((p^1)^\perp \lhd q^1) \otimes ((r^1)^\perp \lhd s^1)) \parr (((q^1)^\perp \lhd r^1) \otimes ((s^1)^\perp \lhd p^1))$

- Moreover, "$\mathrm{BV} \vdash A$?" is NP-complete while "$\mathrm{PL} \vdash A$?" is Σ_2^{p}-complete.

- These logics exemplify two proof-theoretic paradigms going by necessity beyond the sequent calculus: *proof nets* and *deep inference*.
- We realized that the conjecture was very probably false thanks to connections with mainstream graph theory (also initiated by Retoré!).

Backup slide: Reduction CNF-SAT $\to$ elementary round-trip

$$(x \vee y \vee z) \wedge (\neg x \vee y) \wedge (\neg y \vee \neg z)$$

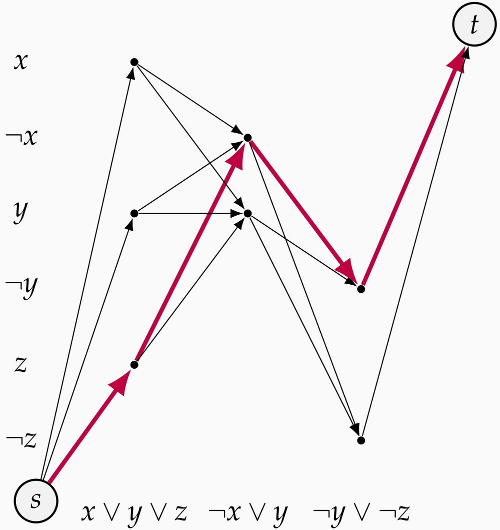
We consider two graphs whose vertices
are the literal occurrences in the clauses:

Backup slide: Reduction CNF-SAT $\to$ elementary round-trip

$$(x \vee y \vee z) \wedge (\neg x \vee y) \wedge (\neg y \vee \neg z)$$

We consider two graphs whose vertices are the literal occurrences in the clauses:

- paths $s \to t$ in 1st graph = choose one literal in each clause

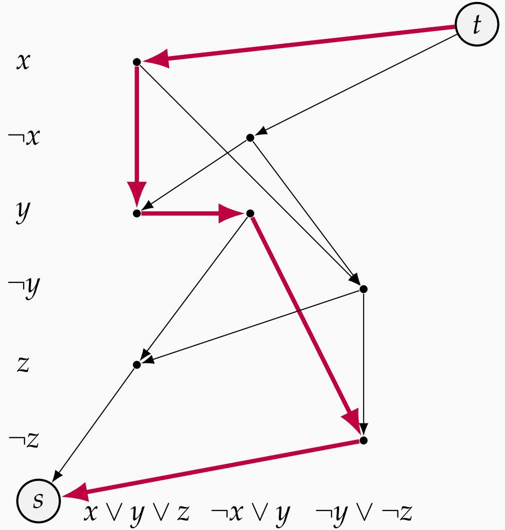


Backup slide: Reduction CNF-SAT $\rightarrow$ elementary round-trip

$$(x \vee y \vee z) \wedge (\neg x \vee y) \wedge (\neg y \vee \neg z)$$

We consider two graphs whose vertices are the literal occurrences in the clauses:

- paths $s \rightarrow t$ in 1st graph = choose one literal in each clause
- paths $t \rightarrow s$ in 2nd graph = choose assignment and visit all $\texttt{false}$ literals (here $x = \texttt{false}$, $y = \texttt{false}, z = \texttt{true}$)



Backup slide: Reduction CNF-SAT $\to$ elementary round-trip

$$(x \vee y \vee z) \wedge (\neg x \vee y) \wedge (\neg y \vee \neg z)$$

We consider two graphs whose vertices are the literal occurrences in the clauses:

- paths $s \to t$ in 1st graph = choose one literal in each clause
- paths $t \to s$ in 2nd graph = choose assignment and visit all `false` literals (here $x =$ `false`, $y =$ `false`, $z =$ `true`)

Non-intersecting pair = satisfying assignment