Implicit automata in typed λ-calculi

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partially based on joint work with Cécilia Pradic (Swansea University)

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Some motivations coming from the λ-calculus

Let's consider the *simply typed λ-calculus* (I assume basic familiarity).

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$\longrightarrow$ typical "power" question: *what* does it compute?

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Some results known for a long time, e.g.

Theorem (Schwichtenberg 1975)

The functions $\mathbb{N}^k \to \mathbb{N}$ definable by simply-typed λ-terms $t : \mathsf{Nat} \to \cdots \to \mathsf{Nat} \to \mathsf{Nat}$ are the extended polynomials (generated by 0, 1, $+$, $\times$, `id` and `ifzero`).

where Nat is the type of *Church numerals*.

Simply typed functions on Church numerals (1)

Recall that the type of Church numerals is $\mathsf{Nat} = (o \to o) \to o \to o$

$$n \in \mathbb{N} \quad \rightsquigarrow \quad \overline{n} = \lambda f.\, \lambda x.\, f\,(\ldots\,(f\,x)\ldots) : \mathsf{Nat} \text{ with } n \text{ times } f$$

- for $n \in \mathbb{N}$, we have $\overline{n} : \mathsf{Nat}$
- conversely $t : \mathsf{Nat} \implies \exists n \in \mathbb{N}.\ t =_{\beta\eta} \overline{n}$

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  $\overline{n} : \mathsf{Nat}[A] = \mathsf{Nat}[A/o] = (A \to A) \to A \to A$
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Question

Choose some simple type A and some term $t : \mathsf{Nat}[A] \to \mathsf{Nat}$.

What functions $\mathbb{N} \to \mathbb{N}$ can be defined this way?

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For $\mathsf{Nat}[A] \to \mathsf{Bool}$, there is a nice characterisation!

Defining languages in the simply typed λ-calculus

Church encodings of binary strings [Böhm & Berarducci 1985]

$\simeq$ `fold_right` on a list of characters (generalizable to any alphabet; $\mathsf{Nat} = \mathsf{Str}_{\{1\}}$):

$$\overline{\texttt{011}} = \lambda f_0.\ \lambda f_1.\ \lambda x.\ f_0\ (f_1\ (f_1\ x)) : \mathsf{Str}_{\{0,1\}} = (o \to o) \to (o \to o) \to o \to o$$

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Simply typed λ-terms $t : \mathsf{Str}_{\{0,1\}}[A] \to \mathsf{Bool}$ define **languages** $L \subseteq \{\mathtt{0}, \mathtt{1}\}^*$

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Example: $t = \lambda s.\ s\ \mathtt{id}\ \mathtt{not}\ \mathtt{true} : \mathsf{Str}_{\{0,1\}}[\mathsf{Bool}] \to \mathsf{Bool}$ (even number of 1s)

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Theorem (Hillebrand & Kanellakis 1996)

All regular languages*, and only those, can be defined this way.*

Regular languages in STLC and implicit complexity

Template for theorems at the structure/power interface

The languages/functions computed by programs of type T in the programming language $\mathcal{P}$ are exactly those in the class $\mathcal{C}$.

(dichotomy taken from Abramsky, cf. e.g. the *Structure meets Power* workshop)

- Hillebrand & Kanellakis: $\mathcal{P}$ = simply typed λ-calculus, $\mathcal{C}$ = regular languages
  - Good news: unlike "extended polynomials", a central object in another field of computer science, namely *automata theory*
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- *Implicit computational complexity*: $\mathcal{C}$ is a complexity class e.g. P, NP, ...
  - ICC has been an active research field since the 1990s (cf. Péchoux's HDR)
  - Historical example (Girard): $\mathcal{P}$ = Light Linear Logic, $\mathcal{C} = \mathsf{P}$ (polynomial time)

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Our “implicit automata” research programme: $\mathcal{C}$ coming from automata theory

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Grandeur et misère de la complexité implicite

Implicit complexity has been very successful in capturing lots of different complexity classes! But the programming languages involved are often ad-hoc…

> *Several systems* […] *have been produced; my favourite being* **LLL**, *light linear logic, which* […] *can harbour all polytime functions. Unfortunately these systems are good for nothing, they all come from bondage: artificial restrictions on the rules which achieve certain effects, but are not justified by use, not even by some natural "semantic" considerations.* — Girard, *From Foundations to Ludics*

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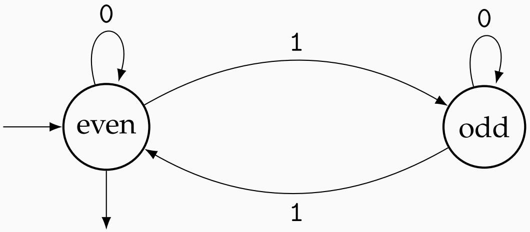
Yet *we didn't ask* for regular languages to appear in the simply typed λ-calculus!

"Implicit automata" challenge: find *natural* characterisations for other automata-theoretic classes of languages/functions using typed λ-calculi

Regular languages

Many classical equivalent definitions (+ STLC with Church encodings!):

- *regular expressions*: `0*(10*10*)*` = "only `0`s and `1`s & even number of `1`s"
- *finite automata* (DFA/NFA): e.g. drawing below



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- *algebraic* definition below (very close to DFA), e.g. $M = \mathbb{Z}/(2)$

Theorem (classical)

A language $L \subseteq \Sigma^*$ *is* regular $\iff$ *there are a* monoid morphism $\varphi : \Sigma^* \to M$ *to a* finite monoid M *and a subset* $P \subseteq M$ *such that* $L = \varphi^{-1}(P) = \{w \in \Sigma^* \mid \varphi(w) \in P\}$.

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Later in the talk: generalizations to *functions* $\Sigma^* \to \Gamma^*$

Before that: proof for languages in STLC

Proof of STLC-definable $\implies$ regular

Theorem (Hillebrand & Kanellakis, LICS'96)

For any type A and any simply typed λ-term $t : \mathsf{Str}_\Sigma[A] \to \mathsf{Bool}$, the language $\mathcal{L}(t) = \{w \in \Sigma^* \mid t\,\overline{w} \to_\beta^* \texttt{true}\}$ is regular.

Part 1 of proof.

Fix type A. Any *denotational semantics* $[\![-]\!]$ quotients words:

$$w \in \Sigma^* \rightsquigarrow \overline{w} : \mathsf{Str}[A] \rightsquigarrow [\![\overline{w}]\!]_{\mathsf{Str}_\Sigma[A]} \in [\![\mathsf{Str}_\Sigma[A]]\!]$$

$[\![\overline{w}]\!]_{\mathsf{Str}_\Sigma[A]}$ determines behavior of w w.r.t. all $\mathsf{Str}_\Sigma[A] \to \mathsf{Bool}$ terms:

$$w \in \mathcal{L}(t) \iff t\,\overline{w} \to_\beta^* \texttt{true} \underbrace{\iff}_{\text{assuming } [\![\texttt{true}]\!] \neq [\![\texttt{false}]\!]} [\![t\,\overline{w}]\!] = [\![t]\!]([\![\overline{w}]\!]) = [\![\texttt{true}]\!]$$

Goal: to decide $\mathcal{L}(t)$, compute $w \mapsto [\![\overline{w}]\!]$ in some denotational model.

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Part 2 of proof.

We use $[\![-]\!] : \mathsf{STLC} \to \mathsf{FinSet}$ to build a DFA with states $Q = [\![\mathsf{Str}_\Sigma[A]]\!]$, acceptation as $[\![t]\!](-) = [\![\texttt{true}]\!]$.

$$\begin{tikzcd} {} \arrow[r] & {[\![\overline{\varepsilon}]\!]} \arrow[r, "\texttt{a}"] & {[\![\overline{\texttt{a}}]\!]} \arrow[r, "\texttt{b}"] & {[\![\overline{\texttt{ab}}]\!]} \arrow[r, "\texttt{b}"] & {[\![\overline{\texttt{abb}}]\!]} \arrow[r] & \cdots \end{tikzcd}$$

$$w \in \mathcal{L}(t) \iff [\![t]\!]\left([\![\overline{w}]\!]_{\mathsf{Str}_\Sigma[A]}\right) = [\![\texttt{true}]\!] \iff w \text{ accepted}$$

$\longrightarrow$ *semantic evaluation* argument (variant: morphism to monoid $[\![\mathsf{Str}_\Sigma[A]]\!]$) $\square$

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$$\longrightarrow [\![\overline{\varepsilon}]\!] \xrightarrow{\mathtt{a}} [\![\overline{\mathtt{a}}]\!] \xrightarrow{\mathtt{b}} [\![\overline{\mathtt{ab}}]\!] \xrightarrow{\mathtt{b}} [\![\overline{\mathtt{abb}}]\!] \longrightarrow \cdots$$

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Usual transduction classes (functions $\Gamma^* \to \Sigma^*$)

What is the right generalization of regular languages to *transductions*?

There are several canonical ones!

$$\underbrace{\text{sequential fonctions}}_{\textit{deterministic}\text{ finite transducers}} \subsetneq \underbrace{\text{rational functions}}_{\textit{nondeterministic}\text{ transducers}} \subsetneq \text{regular functions}$$

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We shall be interested in *regular functions*; many definitions,

such as *streaming string transducers*

Streaming string transducers [Alur & Černý 2010] a.k.a. register transducers

Deterministic finite state automaton + string-valued *registers*. Example:

$$\begin{array}{rrcl} \texttt{mapReverse}: & \{a,b,c,\#\}^* & \to & \{a,b,c,\#\}^* \\ & w_1\#\ldots\#w_n & \mapsto & \texttt{reverse}(w_1)\#\ldots\#\texttt{reverse}(w_n) \end{array}$$

a	c	a	b	$\#$	b	c	$\#$	c	a

$$X = \varepsilon \qquad Y = \varepsilon$$

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Regular functions = computed by copyless SSTs

$a \mapsto \begin{cases} X := aX \\ Y := Y \end{cases} \qquad \# \mapsto \begin{cases} X := \varepsilon \\ Y := YX\# \end{cases}$

each register appears at most once on the right of a := in a transition

Linearity

Regular functions = computed by *copyless* streaming string transducers

Restrictions on "copying power": old theme in automaton theory

Linearity

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Restrictions on "copying power": old theme in automaton theory

λ-calculus counterpart: linear types (Girard 1987)

Here, we use the "$\lambda\ell^{\oplus\&}$-calculus" = *Dual Intuitionistic Linear Logic*
+ additive connectives $\oplus, \&$

(linear = *exactly* once; additives allow us to simulate *at most* once (*affine*))

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$$A, B ::= o \mid \overbrace{\underbrace{A \multimap B}_{\text{linear functions}} \mid A \otimes B}^{\text{multiplicatives}} \mid \overbrace{A \mathbin{\&} B \mid A \oplus B}^{\text{additives}} \mid \underbrace{A \to B}_{\text{non-linear functions}}$$

First steps in implicit transducers

Linear Church encodings

$\mathsf{Str}_{\{a,b\}} = (o \multimap o) \to (o \multimap o) \to (o \multimap o)$, *mutatis mutandis* for Str_Σ

Definition: a type of the $\lambda\ell^{\oplus\&}$-calculus is *purely linear* if it contains no "$\to$"

Theorem (N. & Pradic; proof technique on next slides)

$f : \Gamma^* \to \Sigma^*$ *is* regular $\iff$ $\exists$ *a* purely linear *type* A *and* $t : \mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma$ *in the* $\lambda\ell^{\oplus\&}$*-calculus such that* $\forall w \in \Gamma^*,\ \overline{f(w)} =_\beta t\ \overline{w}$

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- works also for regular tree-to-tree functions
  - similar to "linear high-order tree transducers" [Gallot, Lemay & Salvati 2020] $\to$ characterize reg. tree fn. w/o additives, but with "regular lookahead"
- $\mathsf{Str}_\Gamma[A] \to \mathsf{Str}_\Sigma$ instead $\rightsquigarrow$ *discovery* [N., Noûs & Pradic 2021] of a natural subclass of the *polyregular functions* [Bojańczyk 2018]

Semantic evaluation and categorical automata theory

Again, go from typed λ-terms to automata via *denotational semantics*

- naive semantics of the simply typed λ-calculus in *finite sets*
  $\longrightarrow$ finite automata (Hillebrand & Kanellakis's theorem)

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e.g. $\mathcal{C} = \mathsf{Int}(\mathsf{PFinSet}) \rightsquigarrow$ two-way automata [Hines 2003]
$\longrightarrow$ connection with geometry of interaction, but does not handle additives

Monoidal closed categories vs streaming string transducers

Copyless streaming string transducers $\simeq$ $\mathcal{SR}_\oplus$-automata

- $\mathcal{SR}$ = category of copyless transitions between finite sets of registers
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- copyless SSTs can be determinised [Alur & Deshmukh 2011]
- **our work:** *categorical* determinisation using the existence of some "function spaces" $A \multimap B$ = "partial" monoidal closure!

Monoidal closed structure on the category $(\mathcal{SR}_{\&})_{\oplus}$

For $A, B \in \mathsf{Obj}(\mathcal{SR})$: decompose morphisms

$$\begin{cases} X := abXcY \\ Y := ba \end{cases} \quad \rightsquigarrow \quad \text{shape} \begin{cases} X := Z_1 X Z_2 Y \\ Y := Z_3 \end{cases} \quad + \quad \text{labels } Z_1 = ab, \ldots$$

$$\textit{copyless} \implies \textit{finite} \text{ number of possible shapes} \implies A \multimap B \in \mathsf{Obj}(\mathcal{SR}_{\oplus})$$

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This decomposition is a classical technique, also used to show that comparison-free polyregular functions are closed under composition in our ICALP'21 paper.

Monoidal closed structure on the category $(\mathcal{SR}_{\&})_{\oplus}$

In $(\mathcal{C}_{\&})_{\oplus}$, we have $\bigoplus_u \&_x A_{u,x} \multimap \bigoplus_v \&_y B_{v,y} \;=\; \&_u \bigoplus_v \&_y \bigoplus_x A_{u,x} \multimap B_{v,y}$

Theorem ("Dialectica-like" construction inspired by [Gödel, de Paiva, Hofstra])

Let $\mathcal{C}$ a symmetric monoidal category. If the function space $A \multimap B$ exists in $\mathcal{C}_{\oplus}$ for all $A, B \in \mathsf{Obj}(\mathcal{C})$, then $(\mathcal{C}_{\&})_{\oplus}$ is monoidal closed.

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So, what is known about (compositions of) copyful SSTs?

A robust class of hyperexponential transductions

Macro tree transducers (MTTs) generalize copyful SSTs to trees.

Theorem (Engelfriet & Vogler 1988)

Compositions of MTTs $\iff$ *iterated pushdown transducers*

using stacks of … of stacks of input pointers

$\iff$ *"High level tree transducers"*

$\simeq$ *registers storing* simply typed λ-terms

Trivial observation

This is included in the simply typed λ-definable tree functions.

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But we'll see why the converse might fail, via a detour through *infinite* structures

(subtle restrictions on which λ-terms may appear in registers)

Generating infinite trees

Higher-order pushdown automata = iterated pushdown transducers without input

$$(q_0, [])$$

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$$a$$

$$(q_1, [\,]) \qquad\qquad (q_0, [*])$$

Generating infinite trees

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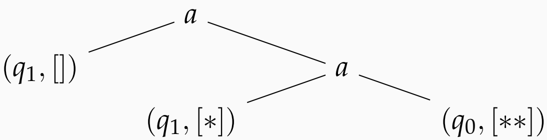

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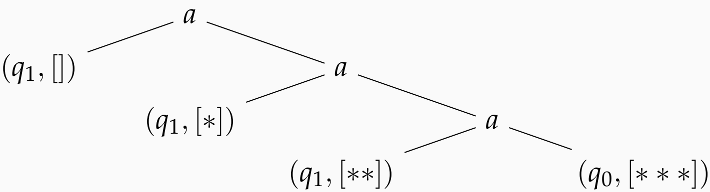

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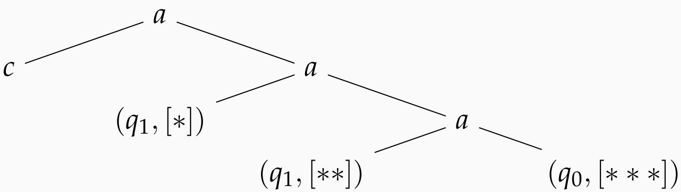

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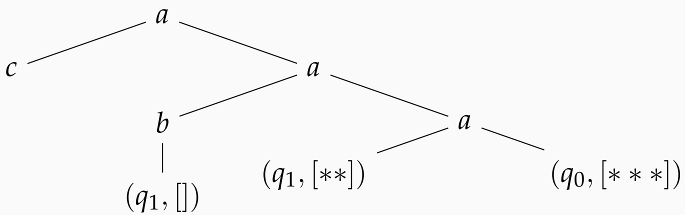

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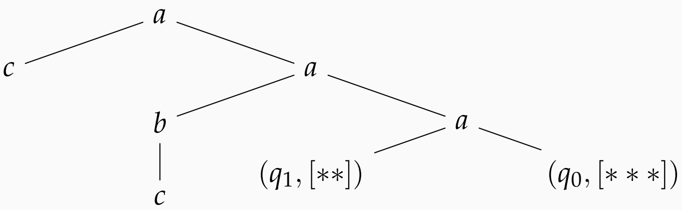

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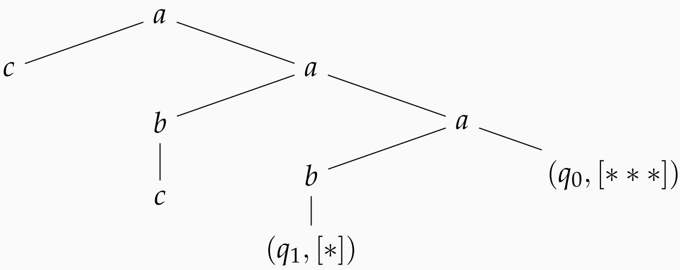

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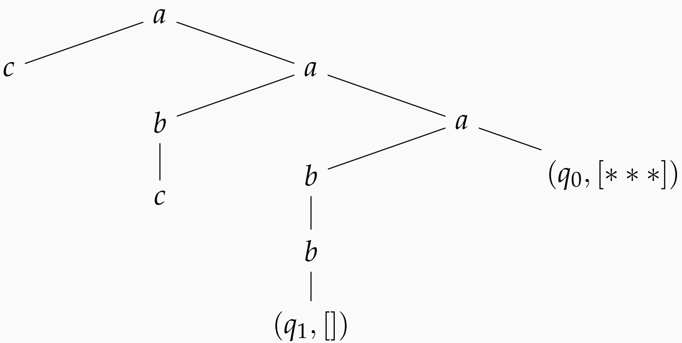

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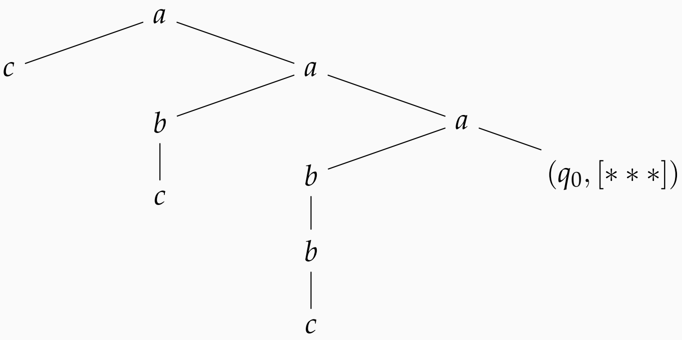

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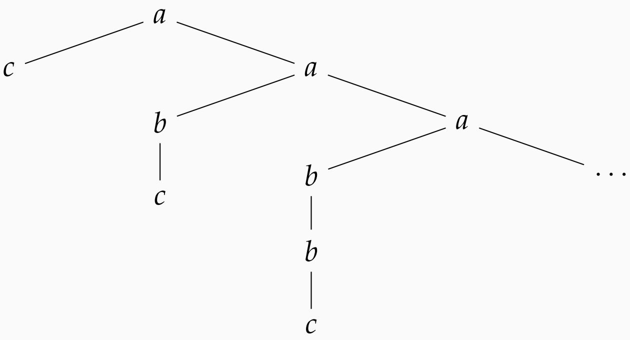

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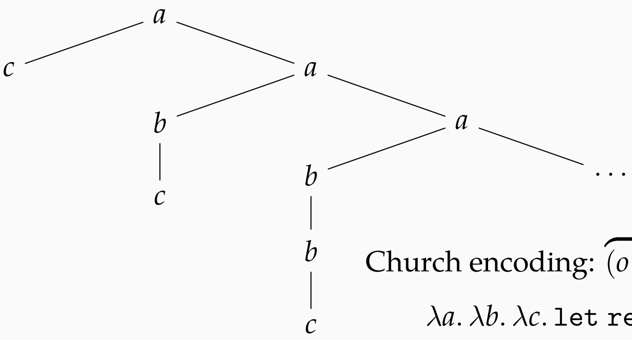

Church encoding: $\overbrace{(o \to o \to o)}^{a} \to \overbrace{(o \to o)}^{b} \to \overset{\overset{c}{\downarrow}}{o} \to o$

$$\lambda a.\, \lambda b.\, \lambda c.\, \texttt{let rec}\, f = \lambda x.\, a\, x\, (f\, (b\, x))\ \texttt{in}\, f\, c$$

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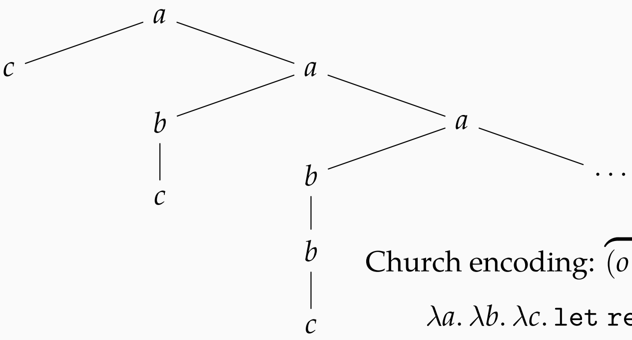

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Theorem (Damm '82; Knapkik, Niwiński & Urzyczyn '02; Salvati & Walukiewicz '12)

HOPDA $\iff$ *so-called* safe *fragment of the simply typed λ-calculus with* `let rec`

Safely λ-definable functions

Equivalence for formalisms generating infinite trees

Higher-order pushdown automata $\iff$ *safe* λ-calculus with `let rec`

- Engelfriet & Vogler's "high level tree transducers" are directly inspired from Damm's work on higher-order grammars $\rightarrow$ implicit safety constraint

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$\rightarrow$ Claim: the following should follow mostly routinely from previous work

Safe λ-terms (w/o `let rec` [Blum & Ong 2009]) of type $\mathtt{Tree}_\Gamma[A] \rightarrow \mathtt{Tree}_\Sigma$ compute the same functions as "high level TTs" / ...

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But some trees can only be generated by *unsafe* recursion schemes [Parys 2012]
$\longrightarrow$ safety could also decrease the λ-definable functions on finite trees

Collapsible pushdown transducers

Theorem (Hague, Murawski, Ong & Serre 2008)

Collapsible PDA *generate the same trees as simply typed λ-terms with* `let rec`

Additional structure on pushdowns of ... of pushdowns + collapse operation

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- Engelfriet & Vogler's proofs rely on inductive characterizations that are not available anymore in this setting…
- Technical issue: "collapsible pushdown transducers" can loop forever,
  the simply typed λ-calculus is terminating

Decomposing the “obvious” theorem: taking divergence into account

Let f : {finite trees} $\rightharpoonup$ {possibly infinite trees} be a partial function.

1. f is computed by a collapsible pushdown transducer
   $\iff$ f is defined by a simply typed λ-term with `let rec`
   $\rightsquigarrow$ straightforward variant of existing proof [Salvati & Walukiewicz 2012]

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   $\rightsquigarrow$ straightforward variant of existing proof [Salvati & Walukiewicz 2012]
2. Furthermore, in that case, there is a simply typed λ-term *without* `let rec` defining a function that coincides with f on $f^{-1}(\{\text{finite trees}\})$
   $\rightsquigarrow$ Plotkin, *Recursion does not always help*, 1982 – arXived in 2022
   again a finite semantics argument! (domain theory)

Conclusion

- "Implicit" characterisations of function classes defined by automata,
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