Coherent interaction graphs: an interactive account of chordless alternating cycles

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"Logic beyond cographs" online seminar, June 16th, 2020

The interactive approach to correctness criteria

- Consider a big space of “paraproofs” (programs)
- Define an “orthogonality relation” by considering the execution/dynamics of paraproofs
- An internal notion of type/formula arises from orthogonality

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We show how cographs and chordless alternating cycles can be obtained this way in an extension of Seiller’s *interaction graphs*.

MLL proofs as matchings

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- First part of the talk: building a denotational semantics
- We start with matchings

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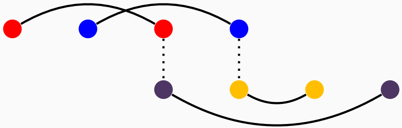

Cut-elimination on matchings

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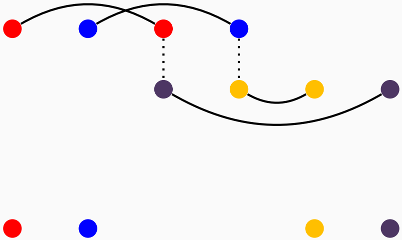

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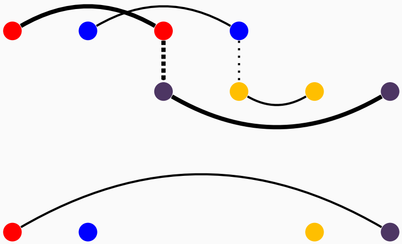

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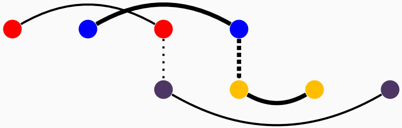

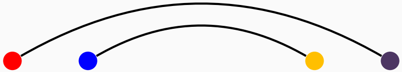

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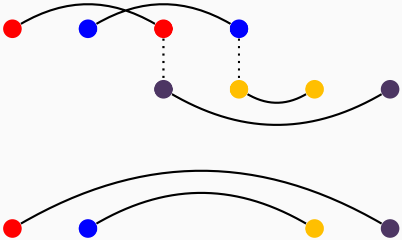

Geometry of Interaction: predict normal form by following paths

Cut-elimination on matchings: another example

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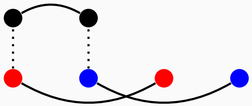

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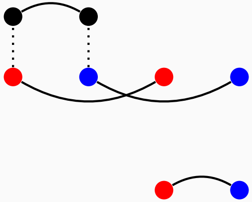

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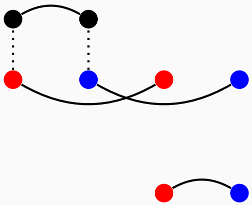

Alternating paths $\simeq$ composition of strategies in game semantics

From matchings to Interaction Graphs

- Matchings are both a GoI and a sort of game semantics
- Execution can be extended to arbitrary graphs:

Definition

Let G, H be two graphs. Their *execution* $G :: H$ is the graph whose vertex set is $V(G) \triangle V(H)$, and whose edges correspond to *alternating paths* between G and H.

$[\![-]\!] : \{\text{MLL proofs}\} \to \{\text{matchings}\} \subset \{\text{graphs}\}$ then enjoys:

Proposition

$[\![\mathbf{cut}(\pi, \rho)]\!] = [\![\pi]\!] :: [\![\rho]\!]$

Interaction graphs as a denotational semantics

Proposition (Associativity / Church–Rosser)

If $V(F) \cap V(G) \cap V(H) = \emptyset$, *then* $(F :: G) :: H = F :: (G :: H)$.

Then it suffices to define types as some sets of graphs with the same vertex set to get a model of MLL (more details later):

Theorem

Interaction graphs constitute a $*$*-autonomous category with composition of morphisms given by execution.*

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- In general, a whole family of models, depending on choices of parameters (e.g. monoid of weights $\to$ quantitative semantics)
- Extension to MELL: generalize from graphs to *graphings* (from measure theory) to represent exponentials

Orthogonality and types (1)

- In the interaction graphs model, morphisms = graphs, objects = **?**
- A set of graphs with the same vertex set...
- ...and the same "specification", think BHK/realizability:
  a proof of A is anything that behaves as prescribed by A
  - Typically we will get $\mathbf{A} \multimap \mathbf{B} = \{F \mid \forall G \in \mathbf{A},\ F :: G \in \mathbf{B}\}$

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- $\to$ types specified by collections of *tests*
- **Tests are also given by graphs**, acting as counter-proofs
- Proofs and counter-proofs related by symmetric *orthogonality* $\perp$

Orthogonality and types (2)

- Morphisms = graphs, objects = **conducts** ($\approx$ "behavior")

Definition

A *conduct* is the orthogonal $T^{\perp} = \{G \mid \forall H \in T, G \perp H\}$ of some set of graphs T (playing the role of tests) over a common vertex set.

- Equivalently: $\mathbf{A}$ is a conduct iff $\mathbf{A}^{\perp\perp} = \mathbf{A}$
- Thus $\mathbf{A}^{\perp}$ can be used as tests for $\mathbf{A}$, and vice versa

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- What is $\perp$? Parameter of the model!
- In general one can define orthogonality as any reasonable predicate on the set of *alternating cycles* between G and H
- This talk: simple choice avoiding technical complications, that also validates the Mix rule

Orthogonality as acyclicity

Definition

$G \perp H \iff \nexists$ alternating cycle between G and H.

Theorem (Adjunction)

If $V(G) \cap V(H) = \emptyset$, then $F \perp (G \sqcup H) \Leftrightarrow (F :: G) \perp H$.

- The adjunction is the key to building a model of MLL: negation is $\mathbf{A} \mapsto \mathbf{A}^\perp$, and $\mathbf{A} \otimes \mathbf{B} = \{G \sqcup H \mid G \in \mathbf{A}, H \in \mathbf{B}\}^{\perp\perp}$
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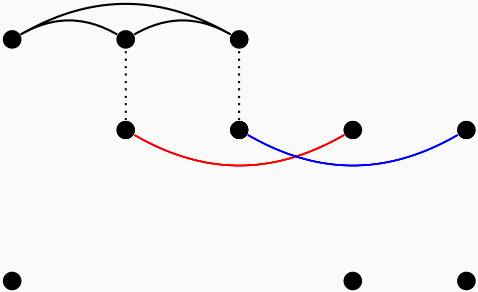
Next: extension to *coherent graphs*

Coherent graphs (1)

Definition

A *coherent graph* is a graph G equipped with a reflexive and symmetric *coherence relation* $\frown_G$ on its edge set $E(G)$.

Example with red $\frown$ black, blue $\frown$ black, red $\not\frown$ blue



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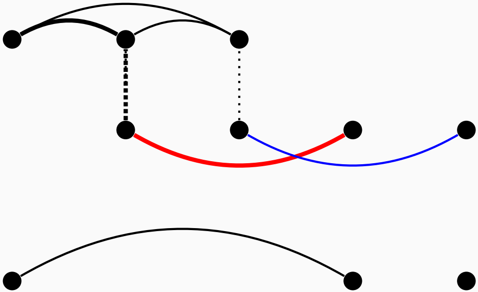

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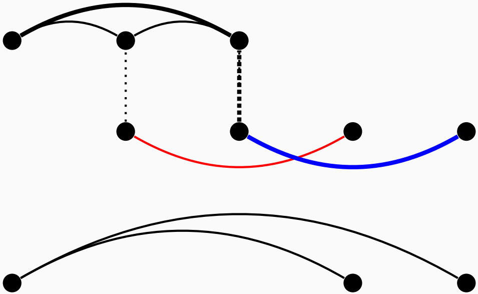

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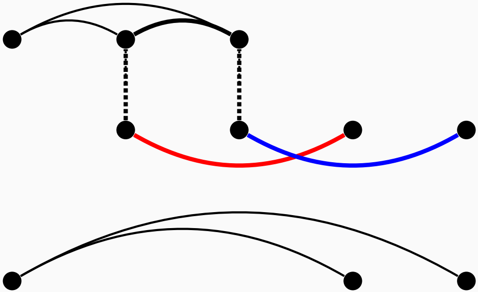

Incoherence: don't take this path

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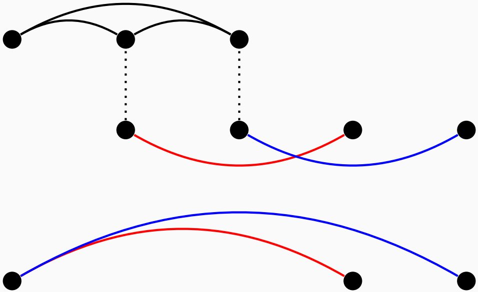

Coherent graphs (2)

- In summary: exec. of coherent graphs = alt. coherent paths (i.e. paths that are *cliques* for $\frown_G$)
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Also, think of a coherent graph (V, E) as the formal sum

$$\sum_{C \subset E} (V, C) \quad (C \text{ clique})$$

$\simeq$ non-deterministic superposition of programs

Tests for coherent interaction graphs

- Original IGs: to generate a type, many tests may be needed
- Coherent IGs: single test needed, by taking a big sum!

Proposition

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Proposition

Analogously, $\{G\}^{\perp} \otimes \{H\}^{\perp} = \{G \parr H\}^{\perp}$ *where* $G \parr H = (G^c \sqcup H^c)^c$ (*with the right coherence relation*).

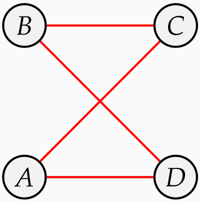
Tests are cographs

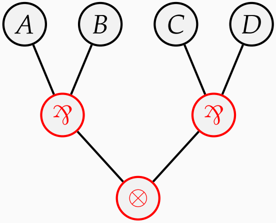
- Formula $F \to$ conduct (w/ atoms sent to $\{*\}$) $\to$ test $T(F)$
  - $T(F)$ generated from $\{*\}$ by $⅋$ and $\sqcup$
- $\mathrm{LCA}_F(A, B)$: least common ancestor of atoms A, B in formula F

Proposition

The underlying graph of $T(F)$ is the cograph *of F:*

- $V(T(F)) = \{\textit{atoms of } F\}$
- $E(T(F)) = \{(A, B) \mid \mathrm{LCA}_F(A, B) = \otimes\}$





Tests are cographs with chordless coherence

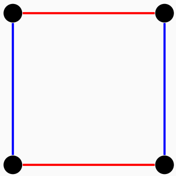
Proposition

For all $e \neq f \in E(T(F))$, $e \not\supset f \iff$

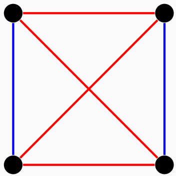
e and f incident or $\exists g \in E(T(F))$ incident to both e and f.

Proposition

Let G, H be graphs and let $\supset_G, \supset_H$ satisfy the above. Then alternating paths/cycles between G and H are coherent iff they are chordless.



Chordless cycle



All cycles have chords

Characterizing denotations of proofs

- Consider a proof π of A
- $[\![\pi]\!] \in [\![A]\!] = \{T(A)\}^{\perp}$, equivalently $[\![\pi]\!] \perp T(A)$
- $\rightarrow$ necessary condition for a graph to come from a proof of A
- Converse?

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Theorem (Reformulation of Retoré 2003)

M perfect matching *and* $M \perp T(A)$
$\iff$ *M comes from a* MLL+Mix *proof of A.*

Corollary (Full completeness)

All perfect matchings *in* $[\![A]\!]$ *come from proofs of A in* MLL+Mix.

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Theorem (Reformulation of Retoré 2003)

M perfect matching *and* $M \perp T(A)$
$\iff$ *M comes from a* MLL+Mix *proof of A.*

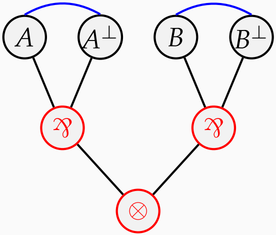
Corollary (Full completeness)

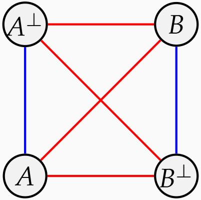
All perfect matchings *in* $[\![A]\!]$ *come from proofs of A in* MLL+Mix.

- paraproofs = all coherent graphs
  a large enough universe to provide good tests
- actual proofs = perfect matchings

Cographic proof nets

- Proof nets = axiom matching + type information
- Traditionally, type tree; but cographs can encode the same thing





Cographic correctness criterion

- Cographic proof structure: (M, G) with $V(M) = V(G)$
  (M matching, G cograph)
- Cographic proof net: is the translation of some sequent proof

Theorem (Retoré 2003 (rediscovered by Ehrhard 2014))

A cographic proof structure (M, G) is a MLL+Mix proof net if and only if there is no chordless alternating cycle between M and G.

- Which we wrote previously as $M \perp G$:
  orthogonality reflects this *correctness criterion*
- Using coherent interaction graphs, we recovered "only if"
- We used "if" – the sequentialization theorem – to deduce our full completeness result

Extension to pomset logic

This part is unpublished (but too trivial to be in a new paper!)

We work with *directed* graphs (everything works the same way...)

Definition

$\mathbf{A} \triangleleft \mathbf{B} = \{G \sqcup H + \text{some dir. edges } \textit{from } V(G) \textit{ to } V(H) \mid G \in \mathbf{A}, H \in \mathbf{B}\}$

Proposition

If $\mathbf{A}, \mathbf{B}$ *are conducts, then* $\mathbf{A} \triangleleft \mathbf{B}$ *is a conduct and* $(\mathbf{A} \triangleleft \mathbf{B})^\perp = \mathbf{A}^\perp \triangleleft \mathbf{B}^\perp$.

Proposition

$\{G\}^\perp \triangleleft \{H\}^\perp = \{G \triangleleft H\}^\perp$ *where*

$$G \triangleleft H = G \sqcup H + \text{all } \textit{directed edges from } V(G) \textit{ to } V(H)$$

with the chordless coherence relation.

Again, we recover Retoré's correctness criterion for pomset logic!

Geometry of Interaction and correctness criteria

- Traditional correctness criteria for proof nets:
  - Generate set of *switchings* from type tree
  - Test each switching against the axiom matching
- Founding observation of GoI: switchings can be seen as counter-proofs (switchings for $A \simeq$ some paraproofs of $A^{\perp}$)
  - Girard's "Multiplicatives" paper, mentioned at the beginning
- $\rightarrow$ tests for a type = switchings
  - Exponentially many switchings
  - Forgetting they all come from the same concise object
- Coherent IGs: single test $\simeq$ superposition of switchings
  - We recover a notion of proof net from this model