A transducer model for simply typed λ-definability

Lê Thành Dũng (Tito) Nguyễn – inspired by previous joint work with Cécilia Pradic

Updated version of a talk given in 2022 (Marseille, Warszawa, Lyon)

The big picture

Basic motivation: natural questions about the *expressiveness of typed λ-calculi*

(minimalistic functional programming languages)

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New result

Answer an old open problem on the λ-calculus, taking inspiration from

- a bunch of (sometimes old) transducer models $\to$ covered in the talk
- more recent work on higher-order recursion schemes

+ raise some speculative questions in pure automata theory

The λ-calculus and Church encodings

A naive syntactic theory of functions:

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\begin{array}{rcl}
f\,x & \approx & f(x) \\
\lambda x.\ t & \approx & x \mapsto t \\
(\lambda x.\ t)\ u \ \to_\beta \ t\{x := u\} & \approx & (x \mapsto x^2 + 1)(42) = 42^2 + 1
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Idea: $n \in \mathbb{N}$ is encoded as $f \mapsto f \circ \ldots (n \text{ times}) \cdots \circ f$

$$\overline{2} = \lambda f.\, \lambda x.\, f\,(f\,x)$$

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The untyped λ-calculus is Turing-complete

The simply typed λ-calculus

We now consider a *type system*: labeling λ-terms with specifications

$$t : A \to B \quad \approx \quad \text{"}t\text{ is a function from }A\text{ to }B\text{"}$$

Simple types: built using "$\to$" from a base type o

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More generally, $t : \mathsf{Nat} \iff \exists n \in \mathbb{N} : t =_{\beta\eta} \overline{n}$

Simply typed functions on Church numerals (1)

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Theorem (Schwichtenberg 1975)

The functions $\mathbb{N}^k \to \mathbb{N}$ definable by simply-typed λ-terms $t : \mathsf{Nat} \to \cdots \to \mathsf{Nat} \to \mathsf{Nat}$ are the extended polynomials (generated by 0, 1, $+$, $\times$, `id` *and* `ifzero`*).*

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A trick to increase expressive power: for any simple type A, for $n \in \mathbb{N}$,

$$\overline{n} : \mathsf{Nat}[A] = \mathsf{Nat}\{o := A\} = (A \to A) \to A \to A$$

(but in general some inhabitants of $\mathsf{Nat}[A]$ don't represent integers)

Open question

Choose some simple type A and some term $t : \mathsf{Nat}[A] \to \mathsf{Nat}$.
What functions $\mathbb{N} \to \mathbb{N}$ can be defined this way?

Simply typed functions on Church numerals (2)

Open question

Choose some simple type A and some term $t : \mathsf{Nat}[A] \to \mathsf{Nat}$.
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Why is nobody working on this seemingly natural question?

- Apparently, low hopes for a nice answer until now
  - you can express towers of exponentials
  - but not subtraction or equality (Statman 198X)
- Not so important for actual programming language theory
  - analogy: functional analysis for differential equations vs Banach space geometry for its own sake...

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Slogan: the above question is not PL theory, it's automata theory!

Defining languages in the simply typed λ-calculus

Church encodings of binary strings [Böhm & Berarducci 1985]

$\simeq$ `fold_right` on a list of characters (generalizable to any alphabet; $\mathsf{Nat} = \mathsf{Str}_{\{1\}}$):

$$\overline{\texttt{011}} = \lambda f_0.\, \lambda f_1.\, \lambda x.\, f_0\ (f_1\ (f_1\ x)) : \mathsf{Str}_{\{0,1\}} = (o \to o) \to (o \to o) \to o \to o$$

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Theorem (Hillebrand & Kanellakis 1996)

All regular languages, *and only those, can be defined this way.*

Automata theory appears in the simply typed λ-calculus

Theorem (Hillebrand & Kanellakis 1996)

The language $L \subseteq \Sigma^*$ *is* regular $\iff$ *there are a simple type* A *and* $t : \mathsf{Str}_\Sigma[A] \to \mathsf{Bool}$ *such that* $\forall w \in \Sigma^*,\ w \in L \Leftrightarrow t\ \overline{w} =_\beta \texttt{true}$

Corollary

A simply typed λ*-term of type* $\mathsf{Str}_\Gamma[A] \to \mathsf{Str}$ *defined a function* $f : \Gamma^* \to \Sigma^*$ *which is* regularity-preserving*:* $L \subseteq \Sigma^*$ *regular* $\implies f^{-1}(L)$ *regular*

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Another good property: these string-to-string functions are *closed under composition* $\longrightarrow$ we might expect them to correspond to some *transducer* model!

However, these functions can have grow as fast as any tower of exponentials which is rarely the case for transducers (but precedents exist!)

So, we started out with a "strategic retreat"...

Implicit automata in linear logic (j.w.w. Cécilia Pradic)

Problem: the simply typed λ-calculus is "too expressive". Possible solution: use a *linear* type system $\longrightarrow$ restrict duplication, hence limit growth rate

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Automata theory counterpart: various "single use restrictions"

Several machine models for *regular functions* of strings and trees involve such restrictions [Bloem & Engelfriet 2000; Engelfriet & Maneth 1999; Alur & Černý 2010; ...]

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$\longrightarrow$ λ-calculus characterizations of *regular* and *comparison-free polyregular* functions

+ star-free languages / aperiodic reg. fn. via non-commutative types

+ upcoming work on atoms (with Clovis Eberhart)

also relying on a single use restriction [Bojańczyk & Stefański 2020]

Streaming string transducers [Alur & Černý 2010]

DFA + string-valued *registers*. Example:

$$\texttt{mapReverse}: \{a,b,c,\#\}^* \to \{a,b,c,\#\}^*$$
$$w_1\#\dots\#w_n \mapsto \texttt{reverse}(w_1)\#\dots\#\texttt{reverse}(w_n)$$

a	c	a	b	$\#$	b	c	$\#$	c	a

$$X = \varepsilon \qquad Y = \varepsilon$$

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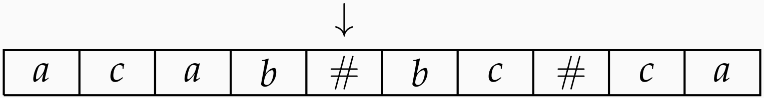

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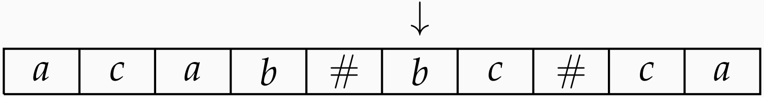

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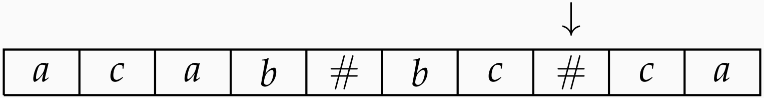

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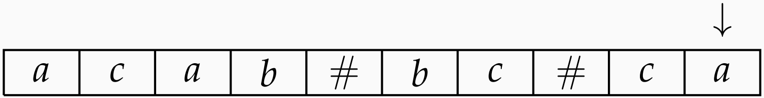

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Regular functions (a.k.a. MSO transductions) = computed by copyless SSTs

$a \mapsto \begin{cases} X := aX \\ Y := Y \end{cases} \qquad \# \mapsto \begin{cases} X := \varepsilon \\ Y := YX\# \end{cases}$

each register appears *at most once* on the right of a := in a transition

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Theorem (Filiot & Reynier 2017)

- *The much older* HDT0L systems *are isomorphic to "simple" copyful SSTs*
- *Copyful SSTs can be simplified* $\rightarrow$ *they compute* HDT0L transductions

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Next, let's search for this keyword in the literature...

Theorem (Ferté, Marin & Sénizergues 2014)

The following compute the same string-to-string functions:

- *another notion of HDT0L transduction* = right-to-left (*simple*) *copyful SSTs*
- level-2 pushdown transducers: *see next slide*

Pushdowns of pushdowns

Theorem (Ferté, Marin & Sénizergues 2014)

Right-to-left (simple) copyful SSTs $\iff$ *level-2 pushdown transducers*

Let's compute $abc \mapsto (c)(bc)(abc)$

$$\begin{bmatrix} \\ \\ \\ [abc] \end{bmatrix}$$

Output:

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Remark: we never need to push sth on the small stacks, they're input suffixes

$\longrightarrow$ "one-way marble" transducers (à la [Douéneau-Tabot, Filiot & Gastin 2020])

Iterated pushdown transducers: using pushdowns of … of pushdowns

We just saw the $k = 1$ case of:

Claim (Sénizergues 2007 — no available proof?)

Composition *of k right-to-left copyful SSTs* $\iff$ *level-$(k+1)$ pushdown transducers*

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Macro tree transducers [Engelfriet & Vogler 1985] can be seen as bottom-up automata with registers, generalizing right-to-left copyful SSTs to trees.

Theorem (Engelfriet & Vogler 1986 (note the different date))

Composition of k macro tree transducers $\iff$ *level-k* (not $k+1$) *pushdown transducers manipulating pointers to the input tree*

(*provide input as pointer to root, not as stack of letters; pointers can only move downwards*)

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Note that this directly generalizes the "one-way marbles" ($k = 1$ on strings)

“Engelfriet’s class” of transductions

In fact, the following are equivalent: [Engelfriet & Vogler ’88; Engelfriet & Maneth ’03]

- Iterated pushdown tree transducers (with pointers)
- Compositions of macro tree transducers
  of attribute grammars a.k.a. tree-walking transducers
  of anything in-between (pebble transducers, MSOT w/ sharing, ...)
- “High level tree transducers”: can be viewed as storing *functions* in registers
  (with subtle restrictions, we’ll come back to that)

A quite robust class of hyperexponential transductions...

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Trivial observation

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Trivial observation

They are included in the simply typed λ-definable functions.

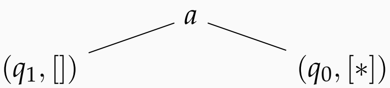
But we’ll see why the converse might fail, via a detour through *infinite* structures

Generating infinite trees

Higher-order pushdown automata = iterated pushdown transducers without input

$$(q_0, [])$$

Higher-order pushdown automata = iterated pushdown transducers without input



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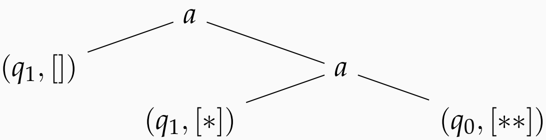

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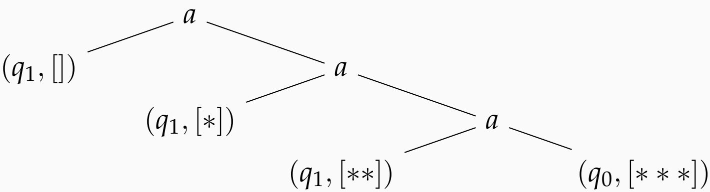

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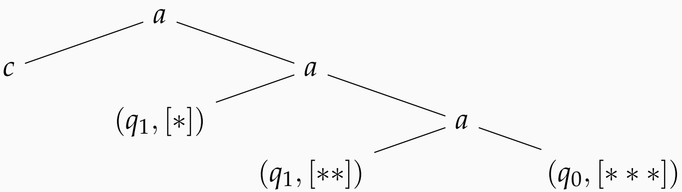

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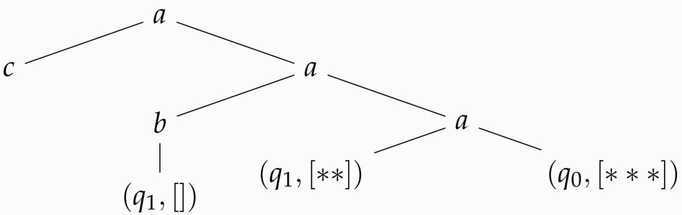

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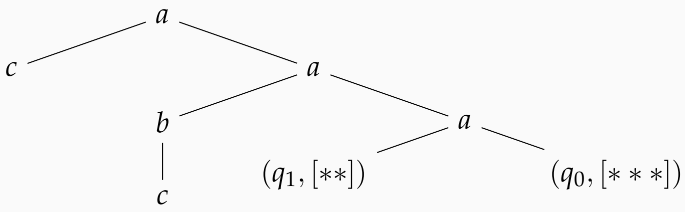

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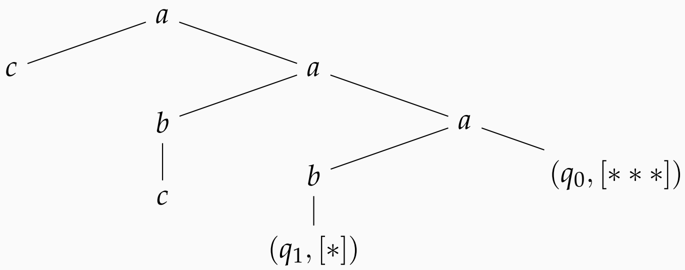

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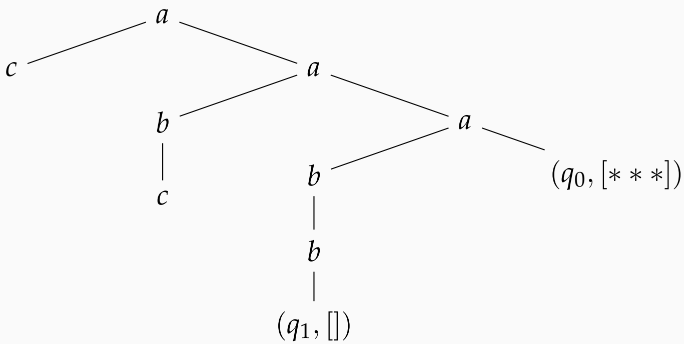

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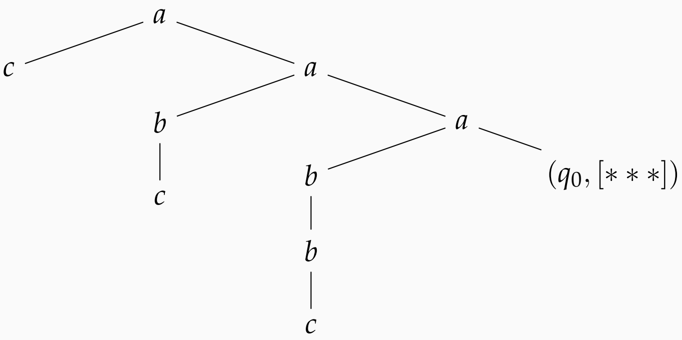

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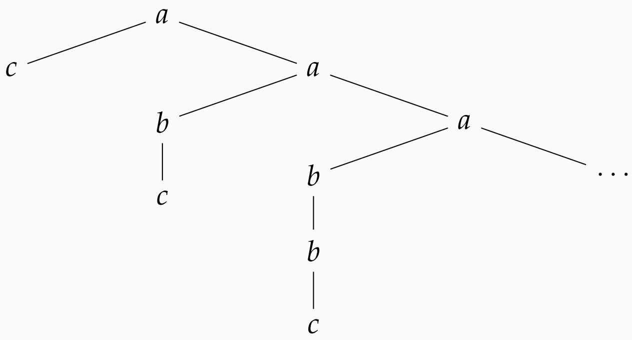

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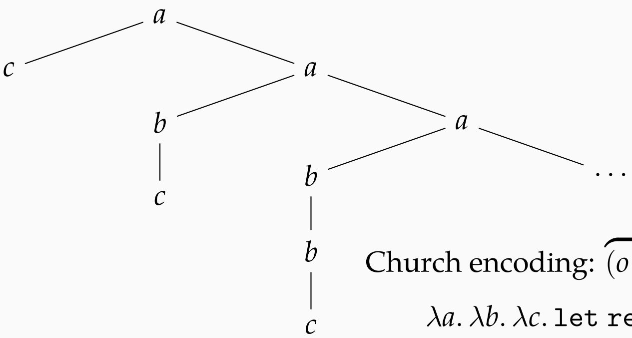

Church encoding: $\overbrace{(o \to o \to o)}^{a} \to \overbrace{(o \to o)}^{b} \to \overset{c}{\overset{\downarrow}{o}} \to o$

$$\lambda a.\, \lambda b.\, \lambda c.\, \texttt{let rec}\, f = \lambda x.\, a\, x\, (f\, (b\, x))\ \texttt{in}\, f\, c$$

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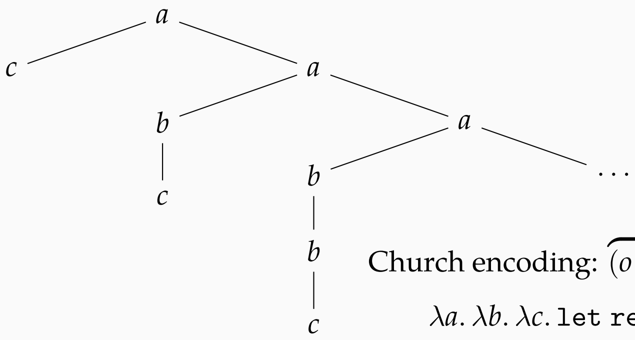

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Theorem (Damm '82; Knapkik, Niwiński & Urzyczyn '02; Salvati & Walukiewicz '12)

HOPDA $\iff$ *so-called* safe *fragment of the simply typed λ-calculus with* `let rec`

Safely λ-definable functions

Equivalence for formalisms generating infinite trees

Higher-order pushdown automata $\iff$ *safe* λ-calculus with `let rec`

- Safety was first introduced in another equivalent formalism, *recursion schemes*
- Engelfriet & Vogler's "high level tree transducers" are directly inspired from Damm's work on safe recursion schemes

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$\rightarrow$ Claim: the following should follow mostly routinely from previous work

Safe λ-terms (w/o `let rec` [Blum & Ong 2009]) of type $\mathsf{Tree}_\Gamma[A] \to \mathsf{Tree}_\Sigma$ compute the same functions as "high level TTs" / iterated pushdown transducers / ...

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But some trees can only be generated by *unsafe* recursion schemes [Parys 2012]
$\longrightarrow$ safety could also decrease the λ-definable functions on finite trees

Collapsible pushdown transducers

Theorem (Hague, Murawski, Ong & Serre 2008)

Collapsible PDA *generate the same trees as simply typed λ-terms with* `let rec`

Additional structure on pushdowns of ... of pushdowns + collapse operation

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The "obvious" theorem

The simply typed λ-definable functions (over Church encodings) are exactly those computable by some "collapsible pushdown tree transducer" model.

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- Engelfriet & Vogler's proofs rely on inductive characterizations that are not available anymore in this setting…
- Technical issue: "collapsible pushdown transducers" can loop forever,
  the simply typed λ-calculus is terminating

Taking divergence into account

Decomposing the "obvious" theorem

Let $f : \{\text{finite trees}\} \rightharpoonup \{\text{possibly infinite trees}\}$ be a partial function.

1. f is computed by a collapsible pushdown transducer
   $\iff$ f is defined by a simply typed λ-term with `let rec`
   $\leadsto$ straightforward variant of existing proof [Salvati & Walukiewicz 2012]

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1. f is computed by a collapsible pushdown transducer
   $\iff$ f is defined by a simply typed λ-term with `let rec`
   $\rightsquigarrow$ straightforward variant of existing proof [Salvati & Walukiewicz 2012]
2. Furthermore, in that case, there is a simply typed λ-term *without* `let rec` defining a function that coincides with f on $f^{-1}(\{\text{finite trees}\})$
   $\rightsquigarrow$ Plotkin, *Recursion does not always help*, 1982 – arXived in 2022!

Open question

Is there some “manifestly total” machine model for these functions?

More questions on simply typed λ-definable functions

- Can they be obtained by composing significantly simpler functions?

  (recall that this works for the safe case i.e. iterated pushdown transducers)
- Does safety harm expressiveness over trees? over strings? over $\{a\}^* \cong \mathbb{N}$?
- Origin semantics using sets of ... of sets of input nodes?

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- Characterizations of subclasses by growth rate?

Theorem (Engelfriet, Inaba & Maneth 2021)

f computed by an iterated pushdown tree transducer $\wedge\ |f(t)| = O(|t|) \iff f$ *is* regular

Conjecture (Maximality of polyregular functions over strings)

f is simply typed λ-definable $\wedge\ |f(w)| = |w|^{O(1)} \iff f$ *is* polyregular

(*i.e. a composition of polynomial growth HDT0L transductions, see* [Bojańczyk 2018])

Conclusion

We started out by studying the functions definable in the simply typed λ-calculus
(on Church-encoded integers/strings/trees, with input type substitution)

- They (strictly?) include most (all?) known transduction classes, while still falling under the scope of automata theory (definable languages are regular)
- We gave a machine model & raised many questions
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Not the first time typed λ-calculi have led us to a new transducer model!

- Most notably, discovery of comparison-free polyregular (or "polyblind") functions, further studied by Douéneau-Tabot [N., Noûs & Pradic 2021]
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