Implicit automata in typed λ-calculi

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joint work with Cécilia Pradic

Updated version of a talk given in 2021–2022

Some motivations coming from the λ-calculus

Let's consider the *simply typed λ-calculus* (I assume basic familiarity).

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Theorem (Schwichtenberg 1975)

The functions $\mathbb{N}^k \to \mathbb{N}$ definable by simply-typed λ-terms $t : \mathsf{Nat} \to \cdots \to \mathsf{Nat} \to \mathsf{Nat}$ are the extended polynomials (generated by 0, 1, $+$, $\times$, `id` and `ifzero`).

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But some open questions have no known satisfying answer...

Simply typed functions on Church numerals (1)

Recall that the type of Church numerals is $\mathsf{Nat} = (o \to o) \to o \to o$

$$n \in \mathbb{N} \quad \rightsquigarrow \quad \overline{n} = \lambda f.\, \lambda x.\, f\,(\ldots\ (f\,x)\ldots) : \mathsf{Nat} \text{ with } n \text{ times } f$$

- for $n \in \mathbb{N}$, we have $\overline{n} : \mathsf{Nat}$
- conversely $t : \mathsf{Nat} \implies \exists n \in \mathbb{N}.\ t =_{\beta\eta} \overline{n}$

Schwichtenberg 1975: $\mathsf{Nat} \to \cdots \to \mathsf{Nat} \to \mathsf{Nat}$ = extended polynomials

Open question

Choose some simple type A and some term $t : \mathsf{Nat}[A] \to \mathsf{Nat}$.
What functions $\mathbb{N} \to \mathbb{N}$ can be defined this way?

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For $\mathsf{Nat}[A] \to \mathsf{Bool}$, there is a nice characterisation!

Defining languages in the simply typed λ-calculus

Church encodings of binary strings [Böhm & Berarducci 1985]

$\simeq$ `fold_right` on a list of characters (generalizable to any alphabet; $\mathsf{Nat} = \mathsf{Str}_{\{1\}}$):

$$\overline{\texttt{011}} = \lambda f_0.\ \lambda f_1.\ \lambda x.\ f_0\ (f_1\ (f_1\ x)) : \mathsf{Str}_{\{0,1\}} = (o \to o) \to (o \to o) \to o \to o$$

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Theorem (Hillebrand & Kanellakis 1996)

All regular languages*, and only those, can be defined this way.*

Regular languages in STλC and implicit complexity

Template for theorems at the structure/power interface

The languages/functions computed by programs of type T in the programming language $\mathcal{P}$ are exactly those in the class $\mathcal{C}$.

(dichotomy taken from Abramsky, cf. e.g. the *Structure meets Power* workshop)

- Hillebrand & Kanellakis: $\mathcal{P}$ = simply typed λ-calculus, $\mathcal{C}$ = regular languages
  - Good news: unlike "extended polynomials", a central object in another field of computer science, namely *automaton theory*
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- *Implicit computational complexity*: $\mathcal{C}$ is a complexity class e.g. P, NP, ...
  - ICC has been an active research field since the 1990s (cf. Péchoux's HDR)
  - Historical example (Girard): $\mathcal{P}$ = Light Linear Logic, $\mathcal{C} = \mathsf{P}$ (polynomial time)

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Our "implicit automata" research programme: $\mathcal{C}$ coming from automaton theory

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"Implicit automata" challenge: find *natural* characterisations for other automata-theoretic classes of languages/functions using typed λ-calculi

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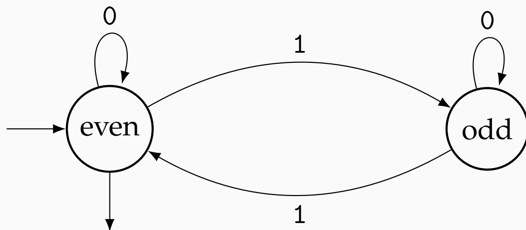
Next: we review regular languages and their generalisations to string *functions*

Regular languages

Many classical equivalent definitions (+ STλC with Church encodings!):

- *regular expressions*: `0*(10*10*)*` = "only `0`s and `1`s & even number of `1`s"
- *finite automata* (DFA/NFA): e.g. drawing below



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Theorem (classical)

A language $L \subseteq \Sigma^*$ is regular $\iff$ *there are a* monoid morphism $\varphi : \Sigma^* \to M$ *to a* finite monoid M *and a subset $P \subseteq M$ such that* $L = \varphi^{-1}(P) = \{w \in \Sigma^* \mid \varphi(w) \in P\}$.

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What is the right generalization to functions $\Gamma^* \to \Sigma^*$ (*transductions*)?

Usual transduction classes (functions $\Gamma^* \to \Sigma^*$)

What is the right generalization of regular languages to *transductions*?

There are several canonical ones!

$$\underbrace{\text{sequential fonctions}}_{\textit{deterministic}\text{ finite transducers}} \subsetneq \underbrace{\text{rational functions}}_{\textit{nondeterministic}\text{ transducers}} \subsetneq \text{regular functions}$$

(“rational languages” = name for regular languages in the French tradition)

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We shall be interested in *regular functions*; many definitions,

such as *streaming string transducers*

Streaming string transducers [Alur & Černý 2010] a.k.a. register transducers

Deterministic finite state automaton + string-valued *registers*. Example:

$$\begin{aligned} \texttt{mapReverse} : \quad \{a,b,c,\#\}^* &\to \{a,b,c,\#\}^* \\ w_1\#\ldots\#w_n &\mapsto \texttt{reverse}(w_1)\#\ldots\#\texttt{reverse}(w_n) \end{aligned}$$

a	c	a	b	$\#$	b	c	$\#$	c	a

$X = \varepsilon \qquad Y = \varepsilon$

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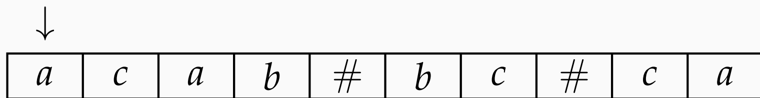

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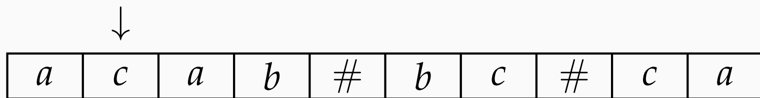

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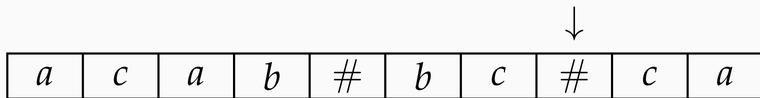

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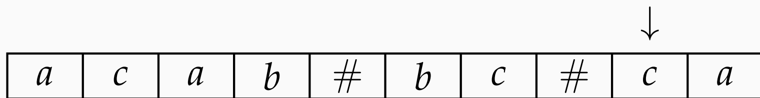

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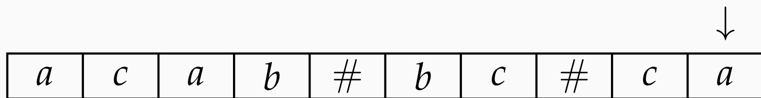

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Regular functions = computed by copyless SSTs

$$a \mapsto \begin{cases} X := aX \\ Y := Y \end{cases} \qquad \# \mapsto \begin{cases} X := \varepsilon \\ Y := YX\# \end{cases}$$

each register appears at most once on the right of a := in a transition

Linearity

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Restrictions on "copying power": old theme in automaton theory

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λ-calculus counterpart: linear types (Girard 1987)

Here, we use the “$\lambda\ell^{\oplus\&}$-calcul” = *Dual Intuitionistic Linear Logic*
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$$A, B ::= o \mid \underbrace{\overbrace{A \multimap B \mid A \otimes B}^{\text{multiplicatives}}}_{\text{linear functions}} \mid \overbrace{A \mathbin{\&} B \mid A \oplus B}^{\text{additives}} \mid \underbrace{A \to B}_{\text{non-linear functions}}$$

First steps in implicit transducers

Linear Church encodings

$\mathsf{Str}_{\{a,b\}} = (o \multimap o) \to (o \multimap o) \to (o \multimap o)$, *mutatis mutandis* for Str_Σ

Definition: a type of the $\lambda\ell^{\oplus\&}$-calculus is *purely linear* if it contains no "$\to$"

Theorem

$f : \Gamma^* \to \Sigma^*$ *is* regular $\iff$ $\exists$ *a* purely linear *type A and t :* $\mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma$ *in the $\lambda\ell^{\oplus\&}$-calculus such that* $\forall w \in \Gamma^*,\ \overline{f(w)} =_\beta t\ \overline{w}$

First steps in implicit transducers

Linear Church encodings

$\mathsf{Str}_{\{a,b\}} = (o \multimap o) \to (o \multimap o) \to (o \multimap o)$, *mutatis mutandis* for Str_Σ

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- Bad news: this class did not already exist in the literature
- Good news: it is nonetheless natural in the context of recent work on *polyregular functions* [Bojańczyk 2018]

A prerequisite to introduce polyregular functions...

Two-way transducers

Finite state + reading head that can move either left or right at each transition

$\longrightarrow$ alternative equivalent definition of *regular functions*

Output:

A prerequisite to introduce polyregular functions...

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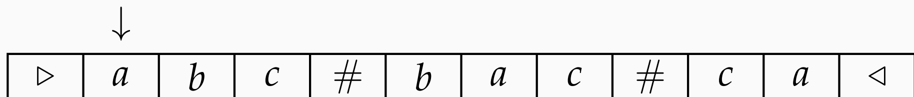

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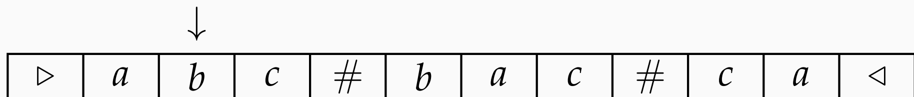

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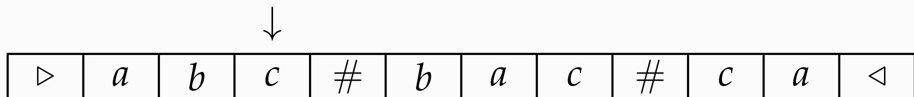

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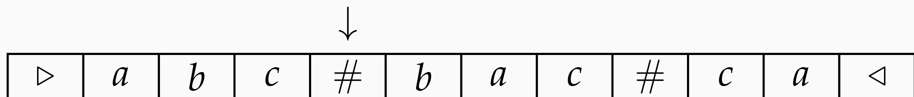

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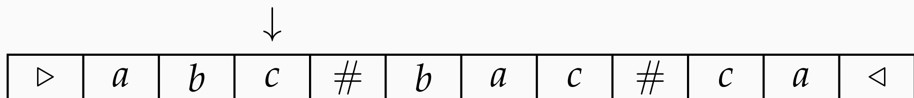

Output: c

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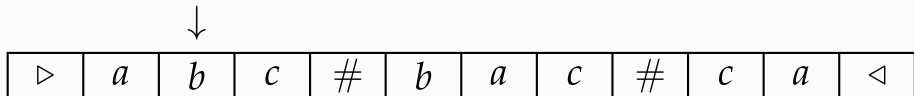

Output: cb

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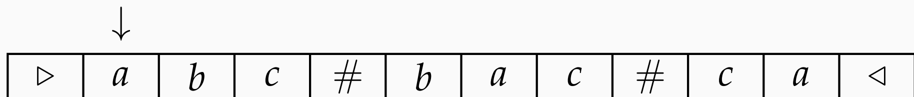

Output: cba

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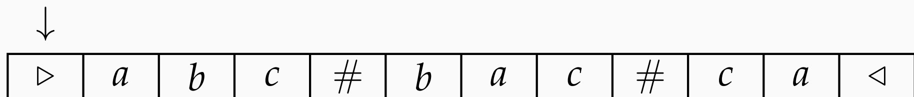

Output: *cba*

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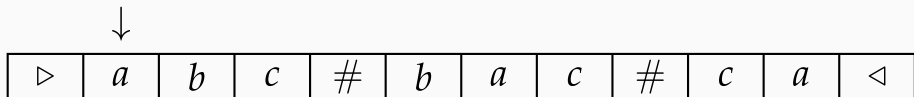

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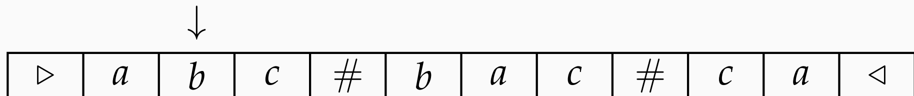

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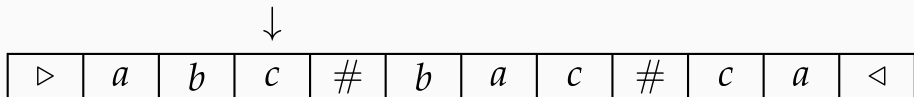

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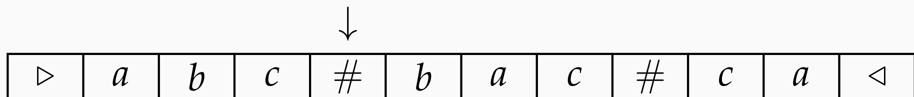

Output: $cba\#$

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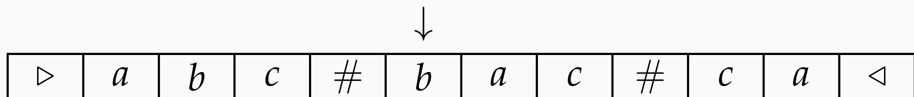

Output: $cba\#$

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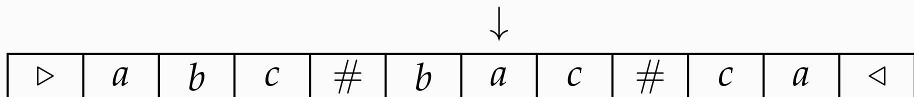

Output: $cba\#$

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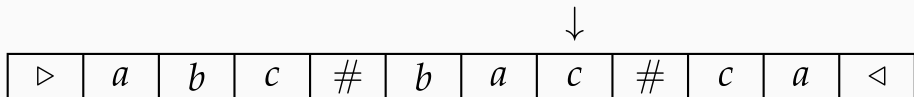

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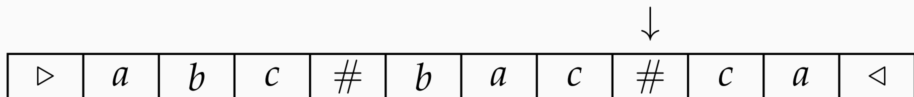

Output: $cba\#$

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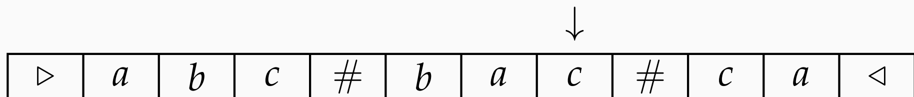

Output: $cba\#c$

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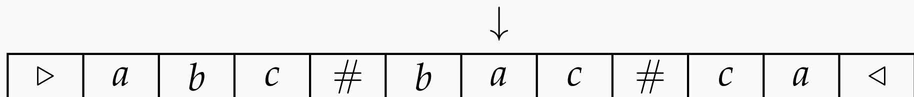

Output: $cba\#ca$

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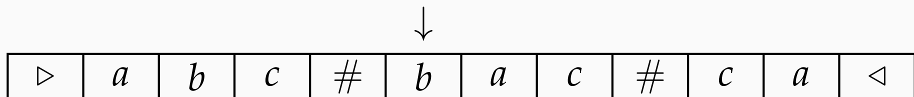

Output: $cba\#cab$

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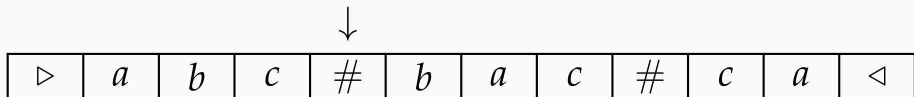

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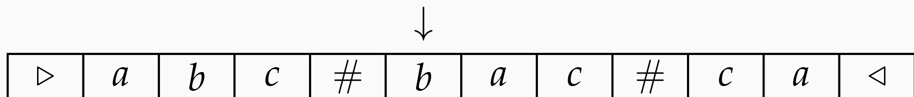

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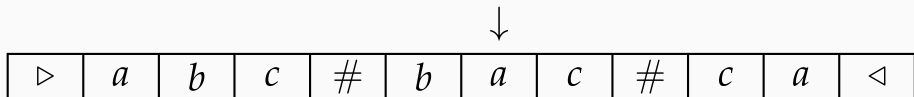

Output: *cba#cab*

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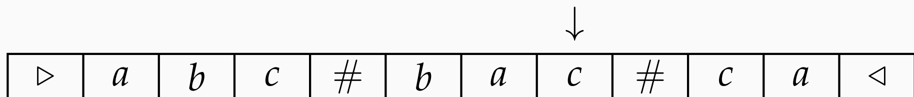

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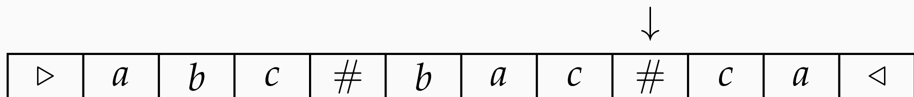

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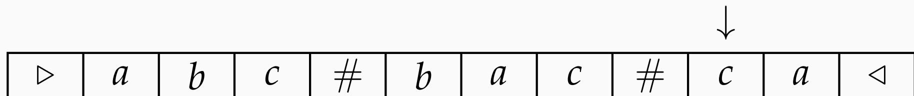

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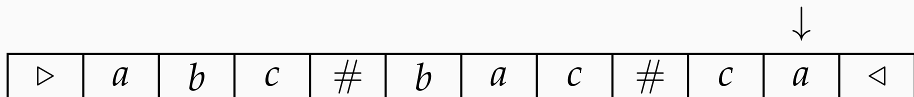

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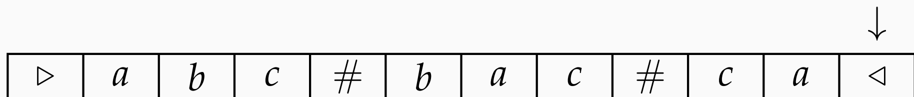

Output: $cba\#cab\#$

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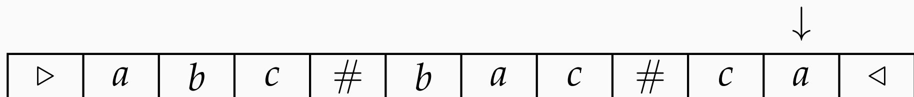

Output: $cba\#cab\#a$

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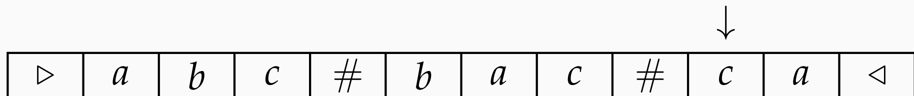

Output: *cba#cab#ac*

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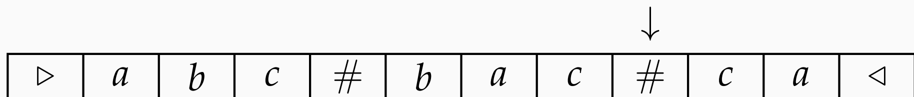

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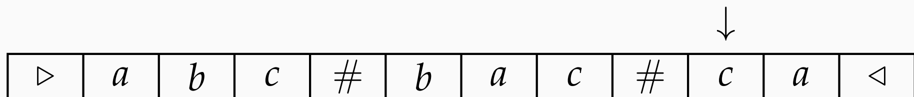

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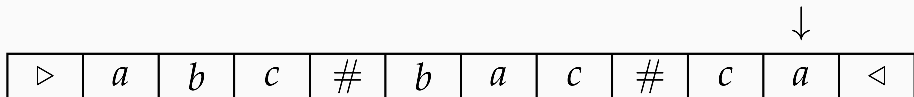

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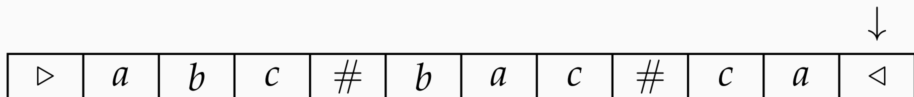

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Remark: they are closely related to the *geometry of interaction* [Hines 2003], which is useful for “implicit automata” (see conclusion)

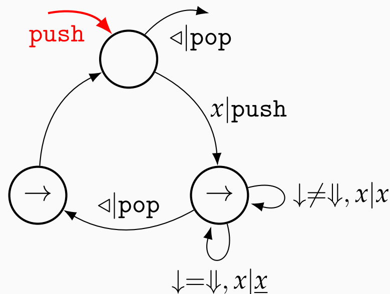
Pebble transducers [Milo, Suciu & Vianu 2000]

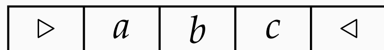
Polyregular functions = computed by <u>k-pebble transducers</u>, $k \in \mathbb{N}$

Finite-state control + *stack* of height $\leqslant k$ of two-way reading heads ("pebbles")

The positions of pebbles can be *compared*. Example:

$$\begin{array}{ccc} \Sigma^* & \to & (\Sigma \cup \underline{\Sigma})^* \\ abc & \mapsto & \underline{a}bca\underline{b}cab\underline{c} \end{array}$$





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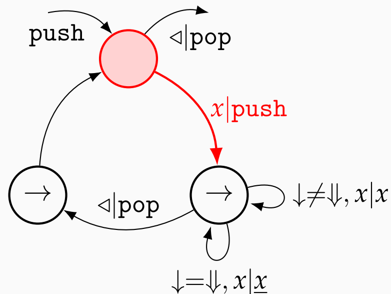

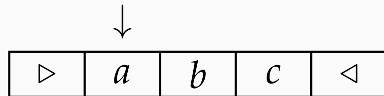

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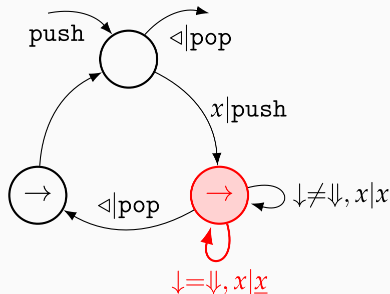

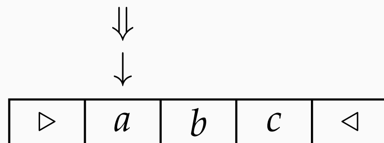

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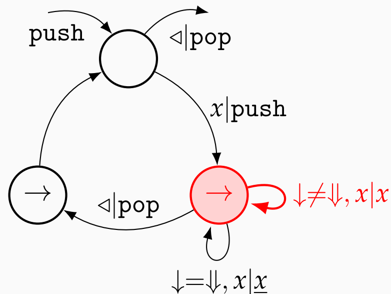

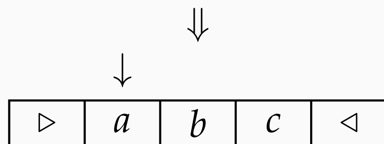

Output: $\underline{a}$

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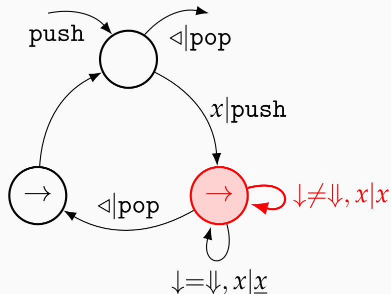

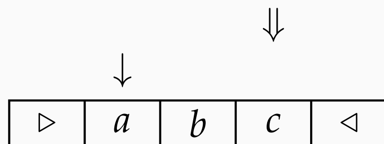

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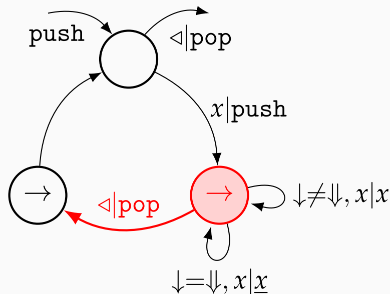

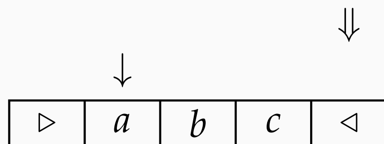

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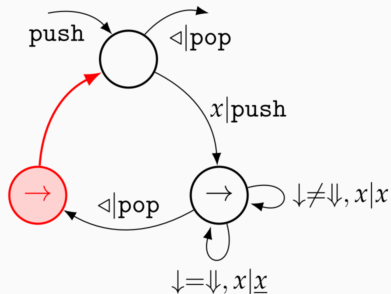

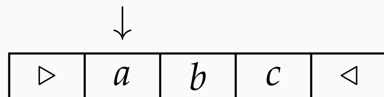

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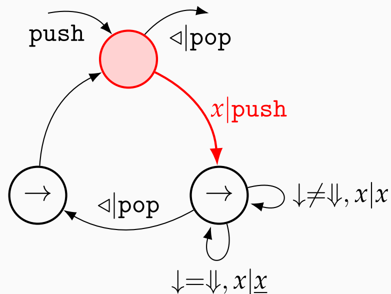

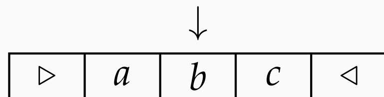

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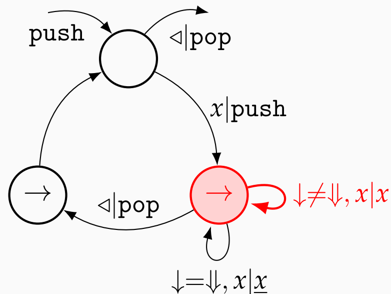

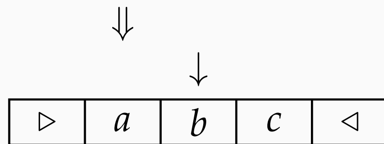

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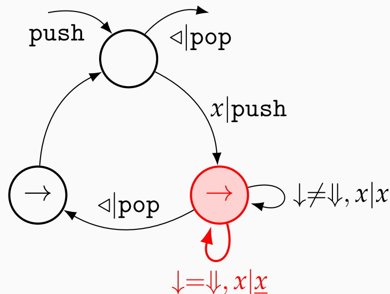

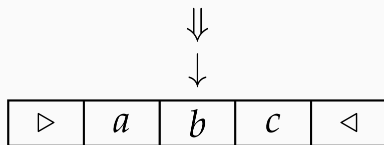

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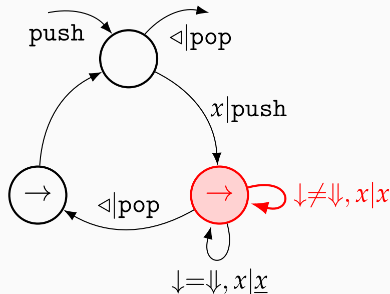

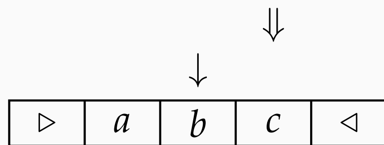

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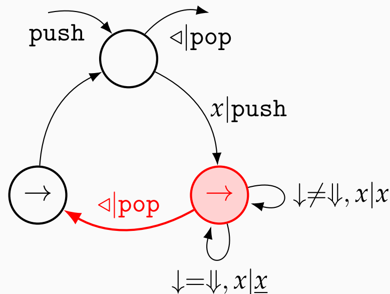

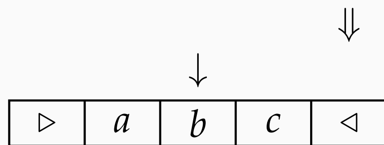

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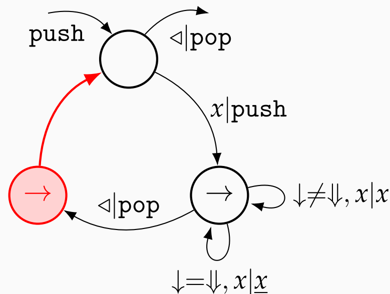

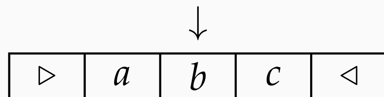

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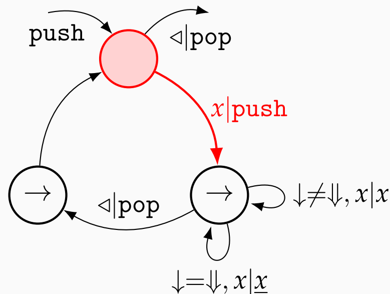

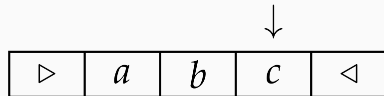

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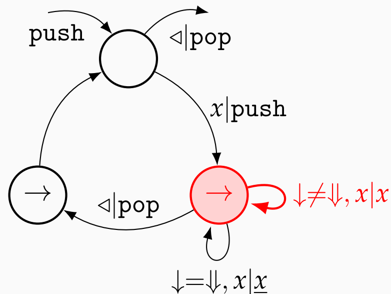

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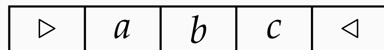

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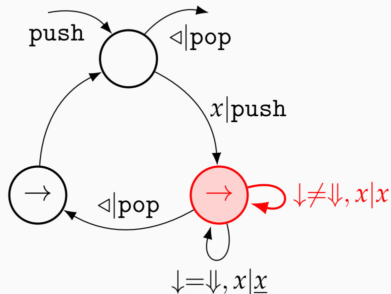

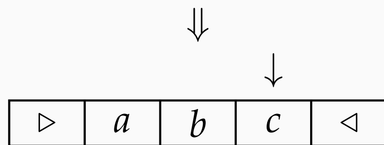

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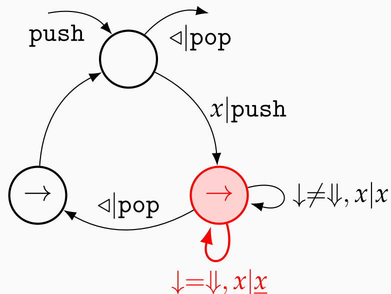

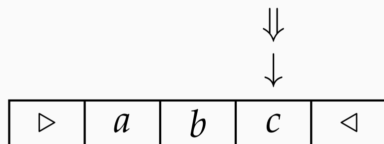

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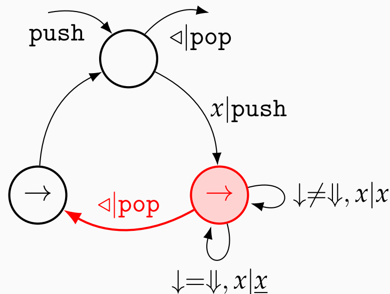

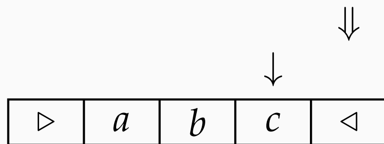

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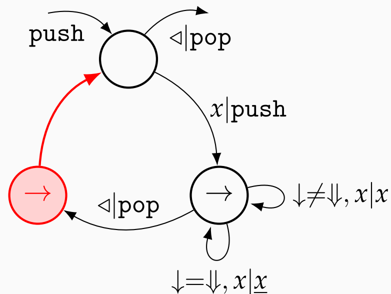

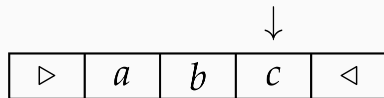

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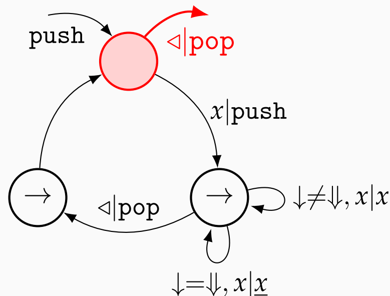

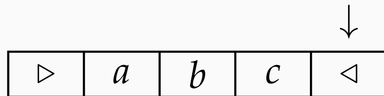

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ICALP'21: we study these functions from a purely automaton-theoretic POV

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  to compensate for the lack of states, we use *decomposition* theorems

Decomposition of regular functions

A compositional "structure" in the study of expressive "power":

Theorem (Bojańczyk et al. (cf. e.g. [Bojańczyk & Stefański 2020]))

A function is regular $\iff$ *it can be obtained as a composition of:*

- sequential *functions, see next slide*
- $\texttt{mapReverse}_\Sigma, \texttt{mapDuplicate}_\Sigma : (\Sigma \cup \{\#\})^* \to (\Sigma \cup \{\#\})^*$ *for* $\# \notin \Sigma$

For $w_1, \ldots, w_n \in \Sigma^*$,

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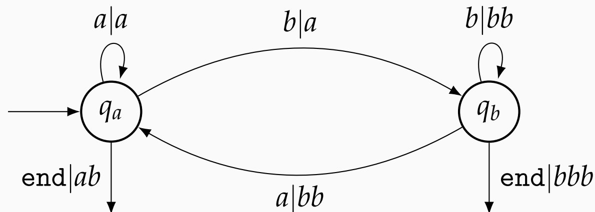
- sequential functions $\to$ correspond to finite states
- `mapReverse` and `mapDuplicate`:

  computed by single-state streaming string transducers

Sequential transducers: an example



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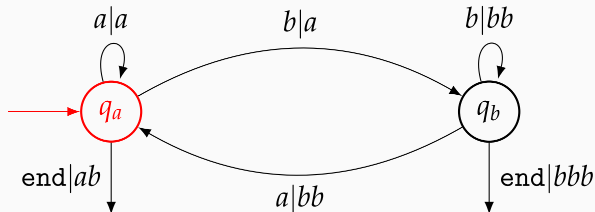
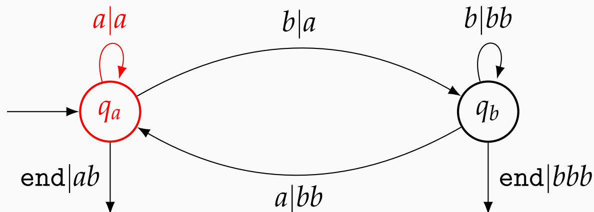


$abba \mapsto$

Sequential transducers: an example



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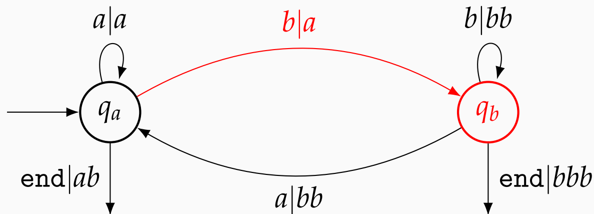

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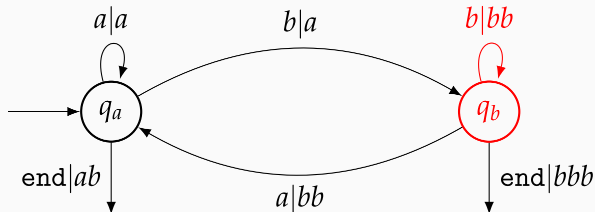

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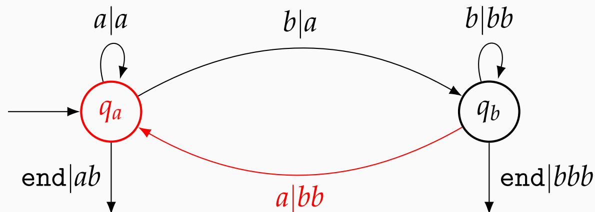

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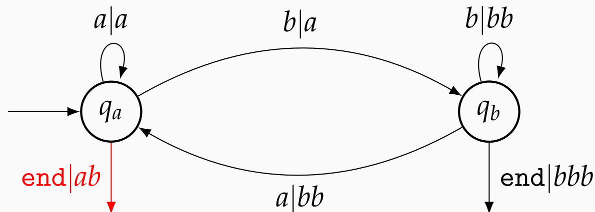

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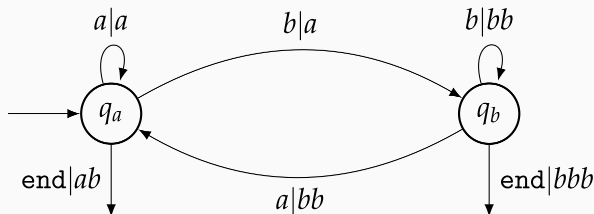

$$abba \mapsto aabbbbab$$

Note: there is an implicit characterisation of sequential functions using *cyclic proofs* [DeYoung & Pfenning, APLAS'16]

The Krohn–Rhodes decomposition

Theorem (Krohn & Rhodes 1965, slightly reformulated)

A function is sequential $\iff$ *it can be obtained as a composition:*

- *of sequential transducers whose transitions* $Q \to Q$ *are bijective;*
- flip-flops *: seq. trans. with 2 states & w/o* $(1 \mapsto 2; 2 \mapsto 1)$ (*example in prev. slide*).

(*Fundamentally, this reflects an algebraic decomposition of finite monoids.*)

You just have to code these “basic blocks” in an *affine* λ-calculus by terms of type $\mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma$… and then those automatically compose!

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Permutations on states are an important primitive

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Connection between Krohn–Rhodes without bijections and *non-commutative* types?

Aperiodicity

Theorem (Krohn & Rhodes 1965, aperiodic case)

A function is aperiodic sequential $\iff$ *it is a composition of "flip-flops".*

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A language $L \subseteq \Sigma^*$ is star-free $\iff$ $\exists$ *a* morphism $\varphi : \Sigma^* \to M$ *to a* finite and aperiodic monoid M *and $P \subseteq M$ such that $L = \varphi^{-1}(P)$.*

(Reminder : w/o "aperiodic" $\rightsquigarrow$ regular languages)

Star-free languages

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Regular expressions w/o (Kleene) *star* – but with *complementation*:

$$L, L' ::= \varnothing \mid \{a\} \mid L \cdot L' \mid L \cup L' \mid L^{\mathrm{c}}$$

for instance $(ab)^* = (b\varnothing^c \cup \varnothing^c a \cup \varnothing^c aa\varnothing^c \cup \varnothing^c bb\varnothing^c)^c$, but $(aa)^*$ is not star-free:
its “minimal monoid” $\mathbb{Z}/2\mathbb{Z}$ is not aperiodic

Implicit automata in non-commutative logic

Star-free regexp: $L, L' ::= \varnothing \mid \{a\} \mid L \cdot L' \mid L \cup L' \mid L^c$

Theorem (N. & Pradic, ICALP 2020)

$L \subseteq \Sigma^*$ *is* star-free $\iff$ $\exists$ *a* purely affine *type* A *and* $t : \mathsf{Str}_\Gamma[A] \multimap \mathsf{Bool}$ *in the "$\lambda\wp$-calcul" such that* $L = \{w \in \Sigma^* \mid t\ \overline{w} =_{\beta\eta} \mathtt{true}\}$

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$\lambda\wp$-calcul : affine variant of *Intuitionistic Non-Commutative Linear Logic*

(Jeff Polakow's PhD thesis, dir. Franck Pfenning)

Idea (going back to Lambek 1958) : the order of arguments matters

"a function $\lambda x.\ \lambda y.\ (\ldots)$ cannot use x *after* y"

$\longrightarrow$ typing judgments $\Phi \mid \Psi \vdash t : A$ for a *set* Φ (non-linear context)

and and *ordered list* Ψ (linear context)

How to go from typed λ-terms to automata?

Semantic evaluation and categorical automaton theory

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Through *denotational semantics* : translation λ-terms $\rightsquigarrow$ algebraic structures

- naive semantics of the simply typed λ-calculus in *finite sets*
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Approach for our "implicit transducer" results

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e.g. $\mathcal{C} = \mathsf{Int}(\mathsf{PFinSet}) \rightsquigarrow$ two-way automata [Hines 2003]
$\longrightarrow$ connection with geometry of interaction, but does not handle additives

Monoidal closed categories vs streaming string transducers

Copyless streaming string transducers $\simeq$ $\mathcal{SR}_\oplus$-automata

- $\mathcal{SR}$ = category of copyless transitions between finite sets of registers
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- copyless SSTs can be determinised [Alur & Deshmukh 2011]
- **our work:** *categorical* determinisation using the existence of some "function spaces" $A \multimap B$ = "partielle" monoidal closure!

Monoidal closed structure on the category $(\mathcal{SR}_{\&})_{\oplus}$

For $A, B \in \mathsf{Obj}(\mathcal{SR})$: decompose morphisms

$$\begin{cases} X := abXcY \\ Y := ba \end{cases} \quad \rightsquigarrow \quad \text{forme} \begin{cases} X := Z_1 X Z_2 Y \\ Y := Z_3 \end{cases} \quad + \quad \text{labels } Z_1 = ab, \ldots$$

copyless $\Longrightarrow$ *finite* number of possible shapes $\Longrightarrow$ $A \multimap B \in \mathsf{Obj}(\mathcal{SR}_{\oplus})$

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$$\textit{copyless} \implies \textit{finite} \text{ number of possible shapes} \implies A \multimap B \in \mathsf{Obj}(\mathcal{SR}_{\oplus})$$

This decomposition is a classical technique, also used to show that comparison-free polyregular functions are closed under composition in our ICALP'21 paper.

Monoidal closed structure on the category $(\mathcal{SR}_{\&})_{\oplus}$

In $(\mathcal{C}_{\&})_{\oplus}$, we have $\bigoplus_u \&_x A_{u,x} \multimap \bigoplus_v \&_y B_{v,y} \;=\; \&_u \bigoplus_v \&_y \bigoplus_x A_{u,x} \multimap B_{v,y}$

Theorem ("Dialectica-like" construction inspired by [Gödel, de Paiva, Hofstra])

Let $\mathcal{C}$ a symmetric monoidal category. If the function space $A \multimap B$ exists in $\mathcal{C}_{\oplus}$ for all $A, B \in \mathsf{Obj}(\mathcal{C})$, then $(\mathcal{C}_{\&})_{\oplus}$ is monoidal closed.

For $A, B \in \mathsf{Obj}(\mathcal{SR})$: decompose morphisms

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- characterisations of transduction classes in a linear λ-calculus
  + star-free languages via a non-commutative type system
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Thanks for your attention!

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