Coherent interaction graphs

A nondeterministic geometry of interaction for MLL

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MLL proofs as matchings (i.e. fixed-point-free involutions)

2 proofs of $A \otimes A \multimap A \otimes A$:

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Cut-elimination on matchings

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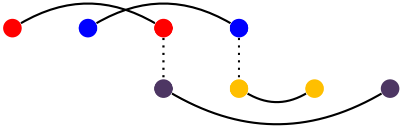

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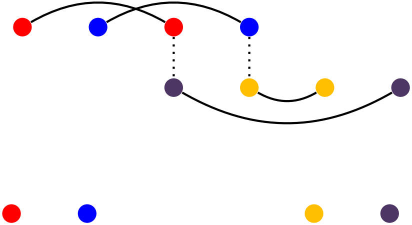

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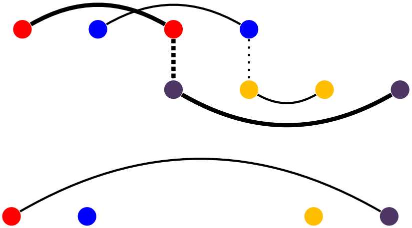

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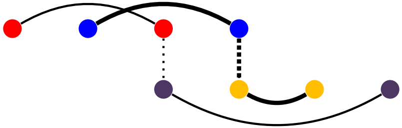

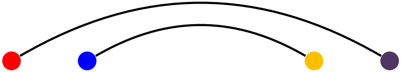

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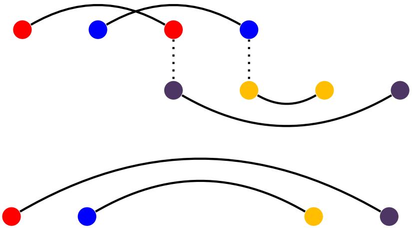
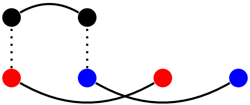


- Geometry of Interaction:
  predict the normal form by following paths

Cut-elimination on matchings: another example

$$\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}}{\vdash A^{\perp} \parr A} \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A} \qquad \dfrac{}{\vdash A^{\perp}, A}}{\vdash A^{\perp} \otimes A^{\perp}, A, A}}{\vdash A \otimes A^{\perp}, A^{\perp} \parr A}}{\vdash A^{\perp} \parr A}\ (\mathrm{Cut})$$



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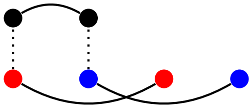

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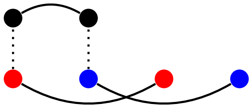

- *Alternating* paths $\simeq$ composition of strategies in game semantics

From matchings to Interaction Graphs

- Matchings are both a GoI and a sort of game semantics
- Execution between matchings can be extended to arbitrary graphs:

Definition

Let G, H be two graphs. Their *execution* $G :: H$ is the graph whose vertex set is $V(G) \triangle V(H)$, and whose edges correspond to *alternating paths* between G and H.

- $[\![-]\!] : \{\text{MLL proofs}\} \to \{\text{matchings}\} \subset \{\text{graphs}\}$ then enjoys:

Proposition

$[\![\mathbf{cut}(\pi, \rho)]\!] = [\![\pi]\!] :: [\![\rho]\!]$

Interaction graphs as a denotational semantics

Proposition (Associativity / Church–Rosser)

If $V(F) \cap V(G) \cap V(H) = \emptyset$, *then* $(F :: G) :: H = F :: (G :: H)$.

- Then it suffices to define types as some sets of graphs with the same vertex set to get a model of MLL, that is:

Theorem

Interaction graphs constitute a $*$*-autonomous category with composition of morphisms given by execution.*

- In general, a whole family of models, depending on choices of parameters (e.g. monoid of weights $\to$ quantitative semantics)
- Extension to MELL: generalize from graphs to *graphings* (cf. Luc Pellissier's talk) to represent exponentials

Our goal: non-determinism / additives

- Let's extend MLL with *non-deterministic* sums of (sub-)proofs:

$$\frac{\vdash \Gamma \quad \dots \quad \vdash \Gamma}{\vdash \Gamma}\ (\textsc{Sum})$$

- How to interpret this rule in interaction graphs?
- Also relevant for *additives*: &-intro is a non-det. superposition
- Formal sums of graphs $\rightarrow$ size explosion

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- A solution: *coherent interaction graphs*
  - Originally introduced in Seiller's PhD for a different purpose
- Using a *coherence relation* is common for additives, e.g. conflict nets (Hughes–Heijltjes), Girard's "Transcendental syntax 2", etc.
  - But we won't treat additives here: technical issues common to all GoI approaches

Coherent graphs

Definition

A *coherent graph* is a graph G equipped with a coherence relation $\frown_G$ on its edge set $E(G)$.

- i.e. $(E(G), \frown_G)$ is a coherent space (which we'll identify with $E(G)$)

Definition

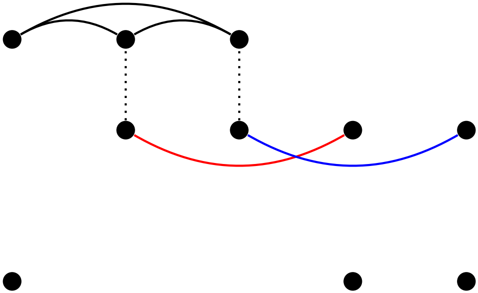
If $V(G) = V(H) = V$, then the *incoherent sum* of G and H is defined as $G \smile{+} H = (V, E(G) \oplus E(H))$. ($\oplus$: disjoint union of coherent spaces)

- $\smile{+}$ interprets the Sum rule
- Think of a coherent graph (V, E) as the formal sum

$$\sum_{C \subset E} (V, C) \quad (C \text{ clique})$$

Execution of coherent graphs: example

- Here red $\frown$ black, blue $\frown$ black, red $\smile$ blue



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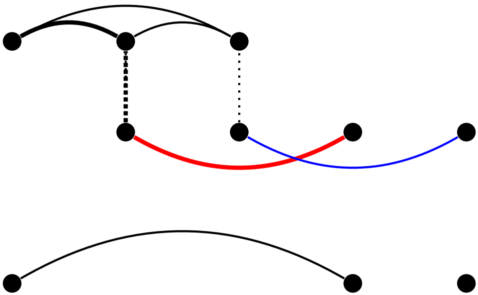

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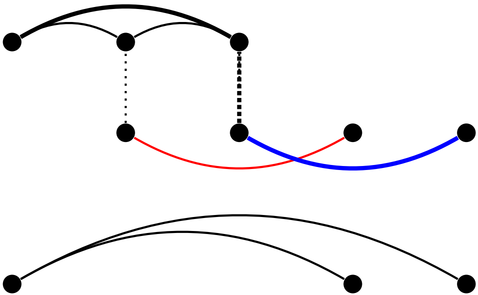

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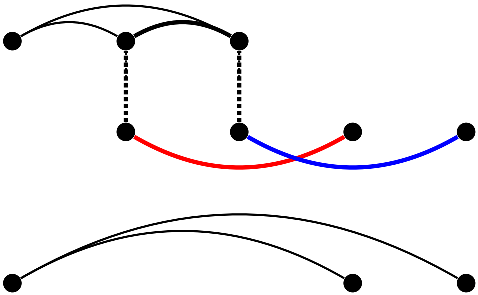

Incoherence: don't take this path

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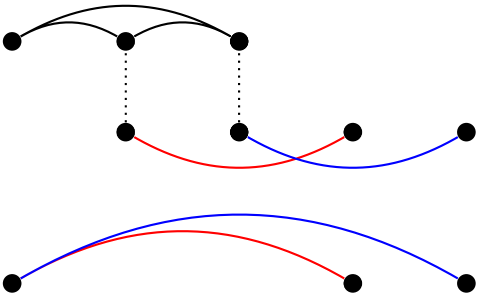

Execution of coherent graphs

- In summary: exec. of coherent graphs = alt. coherent paths

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Theorem

Coherent interaction graphs constitute a $*$-autonomous category with composition of morphisms given by execution.

- Next: a different application of coherent graphs…
  - …namely internalization of a *correctness criterion*
  - We need to present more details on the interpretation of types first

Orthogonality and types (1)

- In the interaction graphs model, morphisms = graphs, objects = **?**
- A set of graphs with the same vertex set…
- …and the same "specification", think BHK/realisability: a proof of A is anything that behaves as prescribed by A
  - Typically we will get $\mathbf{A} \multimap \mathbf{B} = \{f \mid \forall a \in \mathbf{A},\ f :: a \in \mathbf{B}\}$

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- $\rightarrow$ types specified by collections of *tests*

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- $\rightarrow$ types specified by collections of *tests*
- **Tests are also given by graphs**, acting as counter-proofs
- Proofs and counter-proofs related by symmetric *orthogonality* $\perp$

Orthogonality and types (2)

- Morphisms = graphs, objects = **conducts**

Definition

A *conduct* is the orthogonal $T^{\perp} = \{G \mid \forall H \in T, G \perp H\}$ of some set of graphs T (playing the role of tests) over a common vertex set.

- Equivalently: $\mathbf{A}$ is a conduct iff $\mathbf{A}^{\perp\perp} = \mathbf{A}$
- Thus $\mathbf{A}^{\perp}$ can be used as tests for $\mathbf{A}$, and vice versa

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- What is $\perp$? Parameter of the model!
- In general one can define orthogonality as any reasonable predicate on the set of *alternating cycles* between G and H
- This talk: simple choice avoiding technical complications

Orthogonality as acyclicity

Definition

$G \perp H \Leftrightarrow \nexists$ alternating cycle between G and H.

Theorem (Adjunction)

If $V(G) \cap V(H) = \emptyset$, then $F \perp (G \sqcup H) \Leftrightarrow (F :: G) \perp H$.

- The adjunction is the key to building a model of MLL: linear negation is orthogonal, $\mathbf{A} \otimes \mathbf{B} = \{a \sqcup b \mid a \in \mathbf{A}, b \in \mathbf{B}\}^{\perp\perp}$
  - For other choices of $\perp$, need tweaking for adjunction to hold
- We do get $\mathbf{A} \multimap \mathbf{B} = (\mathbf{A} \otimes \mathbf{B}^{\perp})^{\perp} = \{f \mid \forall a \in \mathbf{A},\ f :: a \in \mathbf{B}\}$

Tests for coherent interaction graphs

- Original IGs: to generate a type, many tests may be needed
- Coherent IGs: single test needed, by taking a big sum!

Proposition

$F \perp G \wedge F \perp H \Leftrightarrow F \perp (G \overset{\smile}{+} H)$

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- But this results in a very big test, not sure what we won…
  - Recall efficiency concern w.r.t. formal sums
- More interestingly, small tests often suffice

Operations on single tests

Proposition

$\{G\}^{\perp} \parr \{H\}^{\perp} = \{G \sqcup H\}^{\perp}.$

Proposition

Analogously, from G and H one can define $G \parr H$ such that

- $\{G\}^{\perp} \otimes \{H\}^{\perp} = \{G \parr H\}^{\perp}$
- $|E(G \parr H)| = |E(G)| + |E(H)| + |V(G)| \cdot |V(H)|$

- All conducts generated from $\{*\}$ by $\otimes$ and $\parr$ admit single tests s.t. $|E| \leq |V|(|V| - 1)/2$
- So by interpreting atoms as $\{*\}$ we can always get small tests…

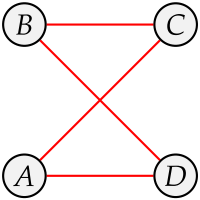
Tests are cographs

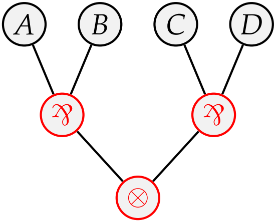
- Formula $F \to$ conduct (w/ atoms sent to $\{*\}$) $\to$ test $T(F)$
  - $T(F)$ generated from $\{*\}$ by $⅋$ and $\sqcup$
- $\mathrm{LCA}_F(A, B)$: least common ancestor of atoms A and B in formula F

Proposition

The underlying graph of $T(F)$ *is the* cograph *of* F*:*

- $V(T(F)) = \{\textit{atoms of } F\}$
- $E(T(F)) = \{(A, B) \mid \mathrm{LCA}_F(A, B) = \otimes\}$





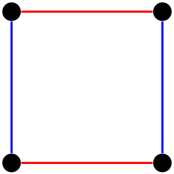
Tests are cographs with chordless coherence

Proposition

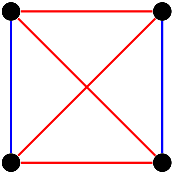
For all $e \neq f \in E(T(F))$, $e \smile f \Leftrightarrow \exists g \in E(T(F))$ *incident to both* e *and* f.

Proposition

Let G and H be coherent graphs s.t. $\frown_G$ *and* $\frown_H$ *satisfy the above. Then alternating paths / cycles between G and H are coherent iff they are* chordless.



Chordless cycle



All cycles have chords

Characterizing denotations of proofs

- Consider a proof π of A
- $[\![\pi]\!] \in [\![A]\!] = \{T(A)\}^{\perp}$, equivalently $[\![\pi]\!] \perp T(A)$
- $\rightarrow$ necessary condition for a graph to come from a proof of A
- Converse?

Characterizing denotations of proofs

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- Converse?

Theorem

M matching *and* $M \perp T(A) \Rightarrow M$ *comes from a* MLL+Mix *proof of* A.

Corollary (Full completeness)

All matchings *in* $[\![A]\!]$ *come from proofs of* A *in* MLL+Mix.

Characterizing denotations of proofs

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Theorem (Reformulation of Retoré 2003 / Ehrhard 2014)

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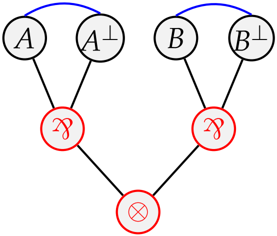
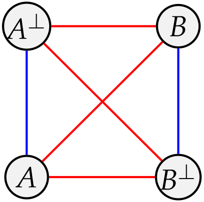
Corollary (Full completeness)

All matchings *in* $[\![A]\!]$ *come from proofs of* A *in* MLL+Mix.

Cographic proof nets

- Proof nets = axiom matching + type information
- Traditionally, type tree; but cographs can encode the same thing





Cographic correctness criterion

- Cographic proof structure: (M, G) with $V(M) = V(G)$ (M matching, G cograph)
- Cographic proof net: is the translation of some sequent proof

Theorem (Retoré 2003 / Ehrhard 2014)

A cographic proof structure (M, G) is a MLL+Mix proof net if and only if there is no chordless alternating cycle between M and G.

- Which we wrote previously as $M \perp G$: orthogonality reflects this *correctness criterion*
- Using coherent interaction graphs, we recovered "only if"
- We used "if" – the sequentialization theorem – to deduce our full completeness result

Geometry of Interaction and correctness criteria

- Traditional correctness criteria for proof nets:
  - Generate set of *switchings* from type tree
  - Test each switching against the axiom matching
- Founding observation of GoI: switchings can be seen as counter-proofs (switchings for $A \simeq$ (kind of) proofs of $A^{\perp}$)
  - Girard's "Multiplicatives" paper
- $\rightarrow$ tests for a type = switchings
  - Exponentially many switchings
  - Forgetting they all come from the same concise object
- Coherent IGs: single test $\simeq$ superposition of switchings
  - We recover a notion of proof net from this model

Conclusion

- Interaction graphs: a graph-theoretic geometry of interaction model (also a primitive game semantics)
- Coherent IGs are *sparse* non-deterministic programs
  - Representation of proofs with formal sums of sub-proofs: *linear* in the size of the proof
  - Tests *quadratic* in the size of the formula
- Future work: additives? MELL? DiLL?
  - Connections with Pagani's visible acyclicity?