Handsome proof nets for MLL+Mix with forbidden transitions

NGUYỄN Lê Thành Dũng
École normale supérieure de Paris
nltd@nguyentito.eu

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Correctness criteria for MLL proof nets: a subject "explored to death[1]"?

- Many correctness criteria already known
- Computational complexity is a solved problem
  - Linear-time algorithms: parsing, dominator tree
  - NL-completeness [Jacobé de Naurois and Mogbil, 2011]
- However, much less is known about MLL with the *Mix rule*
  - A while ago, I asked M. Pagani about references on MLL+Mix proof nets…
  - There is surprisingly little literature on this
  - "it may be much more subtle than expected at first sight"

[1] As aptly remarked by an anonymous reviewer.

Proof nets and algorithmic graph theory

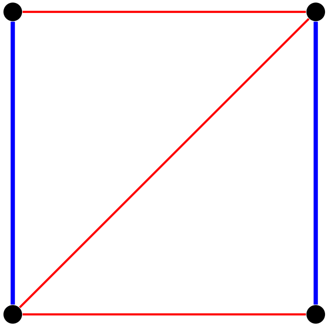
- Why don't we juste use *graph algorithms* to check correctness?
  - Proof nets are graph-like structures
  - Correctness criteria are decision procedures
  - Would let us leverage the work of algorithmists
- Possible answer: the mainstream graph-theoretic toolbox wasn't ready at the birth of linear logic
  - As a result, an idiosyncratic combinatorics developed by the LL community, e.g. paired graphs
- Let us repair this missed opportunity now!
- This will allow us to determine the complexity of deciding correctness for MLL+Mix
  - …and more!

Proof nets and perfect matchings

- In fact, there already is a graph-theoretic correctness criterion, from the article *Handsome proof nets: perfect matchings and cographs* [Retoré, 2003]
- Reduces correctness for MLL *with Mix* to absence of *alternating cycle* for a perfect matching
- Perfect matchings are a classical topic in graph theory and combinatorial optimisation
- Let us start from this point and dig deeper

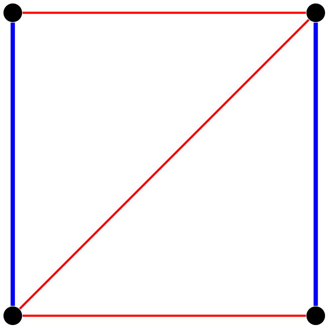
Perfect matchings: reminder (1)

- A *perfect matching* is a set of edges in an undirected graph such that each vertex is incident to exactly one edge in the matching
- Example below: blue edges form a perfect matching



Perfect matchings: reminder (2)

- An *alternating path* is a path
  - without vertex repetitions
  - which alternates between edges inside and outside the matching
- Analogous notion of *alternating cycle*
- $\exists$ alternating cycle $\Leftrightarrow$ the perfect matching is not *unique*



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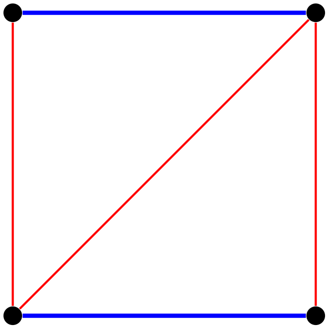
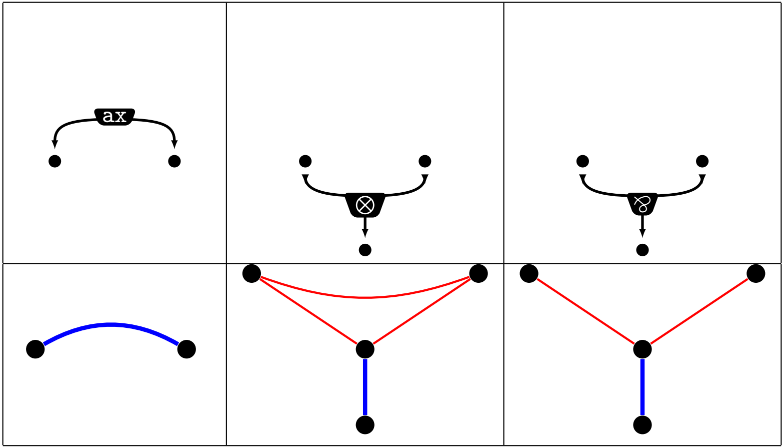


Retoré's *R&B-graphs*



- Correctness criterion: matching is unique, i.e. no alternating cycle

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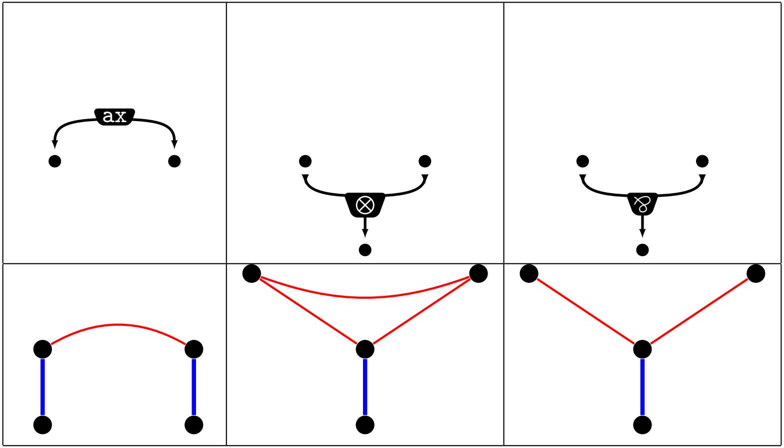
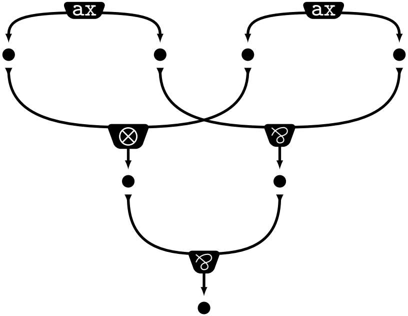


- Correctness criterion: matching is unique, i.e. no alternating cycle
- With this tweak, the matching edges are in bijection with the formulae of the proof structure

R&B-graphs: example (1)



R&B-graphs: example (2)

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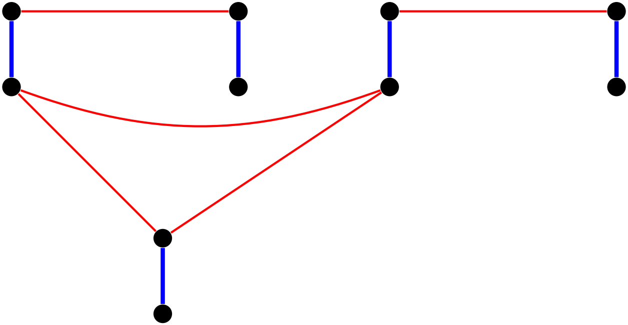

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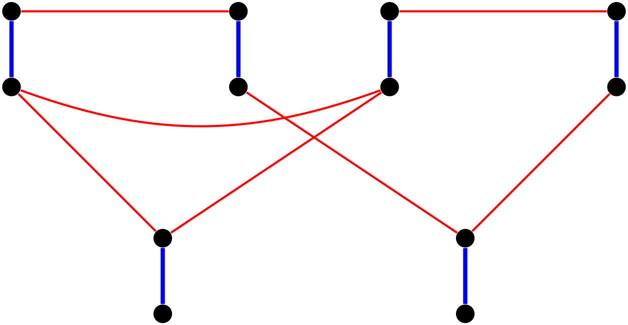

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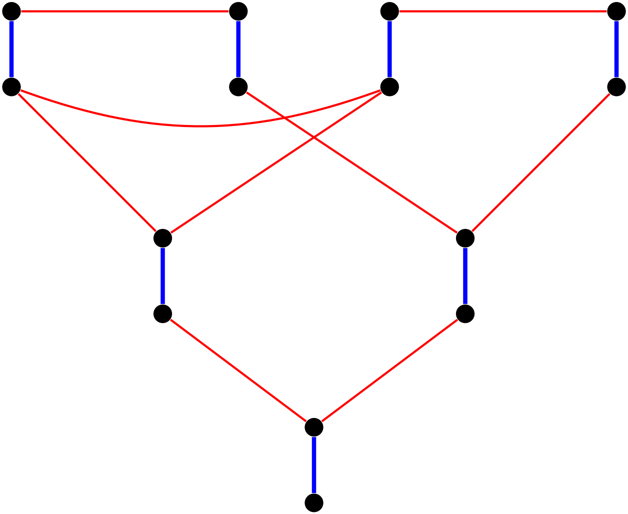
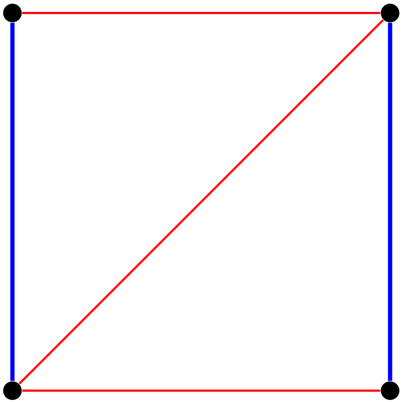


Immediate consequences of R&B-graphs

- Alternating cycles for perfect matchings can be found in *linear time* [Gabow et al., 2001]
- ⇒ **Correctness for MLL+Mix can be decided in linear time**
  - First linear-time criterion for MLL+Mix
  - Also works for MLL without Mix (by Euler–Poincaré…), and simpler than other linear-time criteria: graph theory takes care of the difficult parts!
- Also, a *logspace* reduction to the alternating cycle problem
  - What about the converse?

Alternating cycle → MLL+Mix correctness (1)



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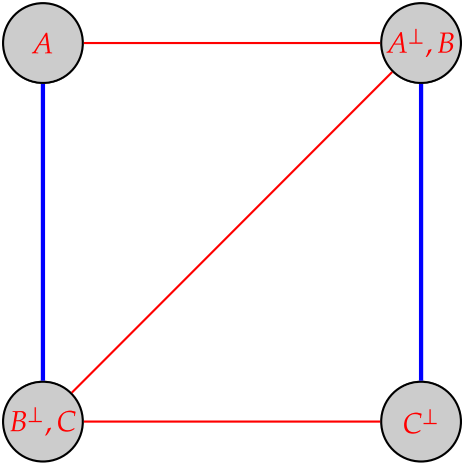

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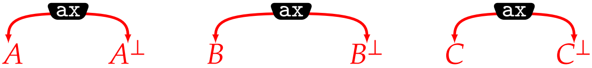

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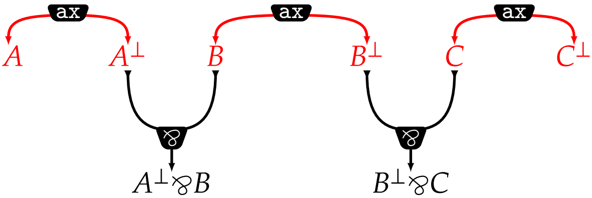

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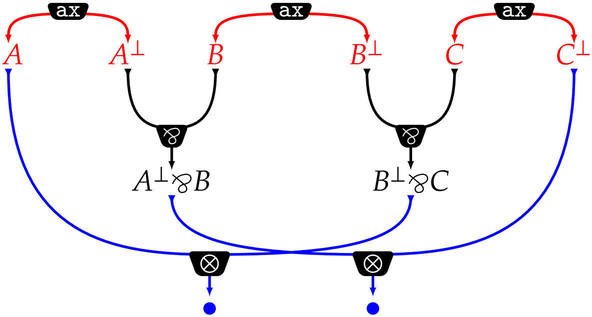

Perfect matchings and sub-polynomial complexity

- Reminder: NC is the class of problem efficiently computable in parallel (polylog(n) time with poly(n) processors)
    - NL $\subseteq$ NC
- Finding an alternating cycle can be done in *randomized* NC (consequence of [Mulmuley et al., 1987])
- *Deterministic* NC? Would solve an open problem from the 80's
- Recently: deterministic *quasi-NC* [Svensson and Tarnawski, 2017]
    - quasipolynomially many processors

On the complexity of MLL+Mix correctness

- Correctness for MLL+Mix is *equivalent* to the alternating cycle problem
- $\Rightarrow$ MLL+Mix correctness $\in$ NL is either **false** or **very hard to prove**
- Contrast with the *NL-completeness* of correctness for MLL
  - Explains why many criteria for MLL, e.g. contractibility, cannot be easily adapted to handle the Mix rule
- Still, **MLL+Mix correctness is in quasi-NC**

Generalizing R&B-graphs to paired graphs

- We can *factorize* Retoré's correctness criterion as a composition of:
  - the Danos–Regnier criterion
  - a purely graph-theoretic construction on *paired graphs*
  - (our tweak on axiom links helps)
- As it turns out, alternating paths in a R&B-graph $\sim$ *trails* not crossing two paired edges *consecutively*
  - A trail may repeat vertices, not edges
  - Not always the same thing as paths in switchings!

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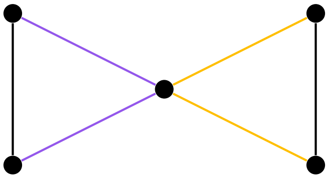

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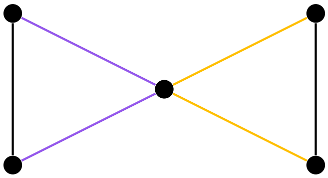
  - Not always the same thing as paths in switchings!
  - But they coincide for paired graphs coming from proof structures



Generalizing even further

- Let's consider paired graphs with *non-disjoint pairs of edges*
- And paths/trails which do not cross paired edges consecutively
  - Pairs are *forbidden transitions*
  - Very general notion of local constraints
- Using R&B-graphs, we can find a *trail avoiding forbidden transitions* between 2 vertices in *linear time*
- **A new**(?) **result in graph theory**
- NP-complete for *paths* avoiding forbidden transitions [Szeider, 2003]
  - (Path: no repeated vertices)

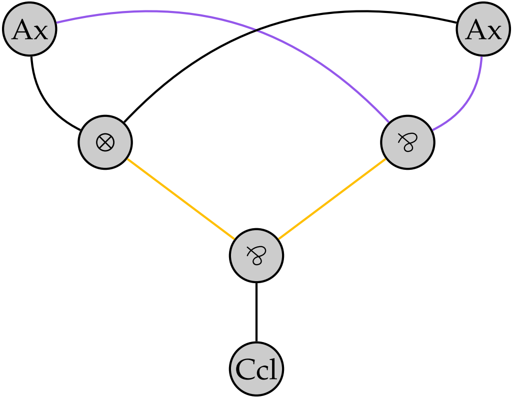
In summary

- An application of graph theory to linear logic: MLL+Mix correctness…
  - can be solved in **linear time**
  - is **probably harder** *(under logspace reductions)* than without Mix
- A result in graph theory taking inspiration from linear logic:
  - an algorithm for finding trails avoiding forbidden transitions
- Hopefully the start of fruitful interactions between these domains!

More stuff I could not talk about

- A graph-theoretic rephrasal of *contracbtility* and *parsing* criteria, in terms of *rainbow paths* in *edge-colored graphs*
  - And edge-colored graphs are related to forbidden transitions…
  - Preprint with all the graph theory stuff coming soon
- An polynomial-time algorithm for computing the *dependency graph* of [Bagnol et al., 2015], and thus the *order of introduction* of links in a proof net
  - Straightforward application of matching theory
  - Relies crucially on the acyclicity property
- A new correctness criterion for proof nets represented as *cographs* [Retoré, 2003] [Ehrhard, 2014]

If time permits…



- A Danos–Regnier paired graph

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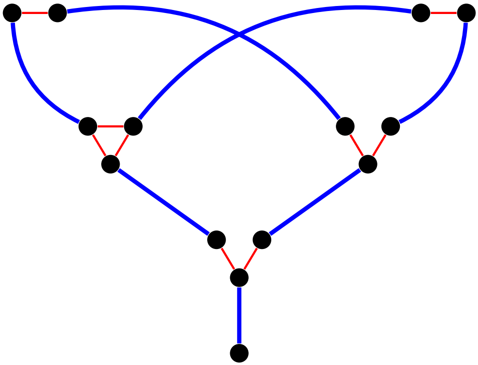

- *Isomorphic* to the R&B-graph seen earlier

If time permits…

- Recipe: take the graph with forbidden transitions and
  - turn edges into matching edges
  - turn vertices into *cliques* outside the matching
  - delete non-matching edges corresponding to forbidden transitions (here, paired edges)
- This construction is actually related to a reduction from *properly colored paths* in *2-edge-colored graphs* to alternating paths in perfect matchings

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Finding paths in graphs avoiding forbidden transitions.