Handsome proof nets for MLL+Mix with forbidden transitions

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Proof nets are a representation of linear logic proofs as *graphs*, instead of trees as in sequent calculus. One drawback of proof nets is that checking the correctness of a proof is non-trivial: there are *proof structures* which locally look like proof nets, but are not correct proofs. One would expect graph algorithms to be of use in solving this problem; unfortunately, correctness criteria are often formulated with structures such as *paired graphs*, which, despite having a purely graph-theoretic definition, are not commonly studied in mainstream graph theory.

To bridge this divide, we show how Retoré's reduction [Ret03] of correctness to the classical and well-studied problem (see [GKT01, HMT]) of *uniqueness of a given perfect matching* (abbreviated to UGPM) relates to paired graphs. We also apply several algorithmic and complexity-theoretic results on matchings to linear logic.

The Mix rule For *Multiplicative Linear Logic* (MLL) proof nets, the problem of correctness is well understood: there are *linear-time* algorithms (e.g. [Gue11]), and a *NL-completeness* result [JdNM11]. However, much less is known about MLL with the *Mix rule* [FR94]. In this setting, as we show, the graph-theoretic point of view proves useful, especially since Retoré's criterion works with Mix. For instance, since there is a *linear-time* algorithm for UGPM, a previously unnoticed corollary of [Ret03] is the existence of a *linear-time correctness criterion for MLL+Mix.* As an added benefit, this also yields a criterion for MLL without Mix which is simpler to describe than other linear-time criteria.

Paired graphs vs perfect matchings We observe that paired graphs are a special case of a structure known in the algorithms community: *graphs with forbidden transitions* [Sze03]. This provides a first connection with matchings: finding paths in such graphs is known to reduce (in some cases) to a matching problem.

To a given proof structure, we associate a *correctness graph* with forbidden transitions. One can then recover exactly Retoré's reduction by taking a sort of "line graph" of the correctness graph: this reduction is shown to be essentially the composition of paired-graph-based criteria with a generic construction on graphs with forbidden transitions.

However, as far as we know, the specific "line graph" that we define has not appeared previously in the literature. We show that it is useful for graph algorithms as well: it solves the problem of finding a *trail* avoiding forbidden transitions. (Trails, unlike paths, can have repeated vertices, but they cannot have repeated edges.)

Applications to correctness with Mix This allows us to give a simple formulation of our correctness criterion: *a proof structure is correct (for MLL+Mix) if and only if its correctness graph has no closed trail avoiding forbidden transitions.* This can be tested in linear time.

We also provide a straightforward proof of *sequentialization*, i.e., that every proof structure which is correct according to our criterion is a proof net. Our proof avoids technical difficulties by relying on a structural property of graphs with forbidden transitions. This property generalizes *Kotzig's theorem* on unique perfect matchings, which was already used by Retoré for his own proof of sequentialization.

Complexity considerations We shed light on the relative complexity of the correctness problem for MLL and for MLL+Mix through the connections with graph theory. Although they both have linear time complexity, we show that the latter is unlikely to be in NL: it is *equivalent under logspace reductions* to UGPM and to closed trails avoiding forbidden transitions.

UGPM is not even known to be in NC, the class of problems with efficient *parallel* algorithms, which includes NL. What is known is that if it were in NC, then deciding whether a graph admits a unique perfect matching would also be in NC, solving an open problem posed by László Lovász in the 80's [KVV]. However, recent progress has shown that the existence of a perfect matching can be decided in quasi-NC [ST17], a result which also puts *correctness with Mix in quasi-NC*.

A novel application: computing the order of introduction To provide another application of our techniques, we show that the *order of introduction* of links in a proof net, i.e., the partial order "A must be introduced above B in any sequentialization", can be computed in polynomial time and in quasi-NC. Our algorithm relies on a characterization of this order from [BDS] (which only works on proof nets without Mix), reformulated in terms of paths avoiding forbidden transitions.

Perspectives Our results suggest that *the algorithmic theory of MLL+Mix correctness criteria is the same as the theory of unique perfect matchings.* We hope that this correspondence can be refined to give a graph-theoretical rephrasing of the correctness problem for MLL without Mix. This would allow tools developed for linear logic to be transposed to graph algorithms.

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