(CR15) CATEGORY THEORY FOR COMPUTER SCIENTISTS: HOMEWORK 2

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Exercise 1

We say that a subset $X \subseteq A$ of a set A is *upwards closed* for a preorder $\preceq$ on A when $\forall x \in X, \forall y \in A, x \preceq y \Rightarrow y \in X$. Let $\mathrm{UpClosed}(A, \preceq)$ be the set of all upwards closed subsets of A for $\preceq$. Recall that subsets of A are partially ordered by the inclusion relation $\subseteq$.

Verify that the following data defines a functor F:

$$\begin{aligned} F\colon \mathbf{PreOrd}^{\mathrm{op}} &\to \mathbf{Ord} \\ (A, \preceq_A) \in \mathrm{ob}(\mathbf{PreOrd}^{\mathrm{op}}) &\mapsto (\mathrm{UpClosed}(A, \preceq_A), \subseteq) \\ f \in \mathbf{PreOrd}^{\mathrm{op}}(A, B) &\mapsto (X \in \mathrm{UpClosed}(A, \preceq_A) \mapsto f^{-1}(X)) \end{aligned}$$

Exercise 2

Let a and b be two distinct letters. Let $\texttt{repeat}_a\colon n \in \mathbb{N} \mapsto [a, \dots, a]$ (with n times a) and $\texttt{repeat}_b\colon n \in \mathbb{N} \mapsto [b, \dots, b]$ (with n times b); they are monoid homomorphisms (with respect to addition on $\mathbb{N}$).

1. Let M be a monoid and $(f, g) \in \mathbf{Mon}(\mathbb{N}, M)^2$. Show that there is a unique morphism $h \in \mathbf{Mon}(\{a, b\}^*, M)$ such that $f(1) = h(\texttt{repeat}_a(1))$ and $g(1) = h(\texttt{repeat}_b(1))$. (Hint: universal property of free monoids.)

2. Show that $(\{a, b\}^*, \texttt{repeat}_a, \texttt{repeat}_b)$ is a coproduct of $\mathbb{N}$ with itself in the category of monoids $\mathbf{Mon}$.

Exercise 3

1. Let $\mathcal{C}$ be a category, 1 be a terminal object of $\mathcal{C}$ and $A \in \mathrm{ob}(\mathcal{C})$. Prove that there exist morphisms $\pi_1 \in \mathcal{C}(A, A)$ and $\pi_2 \in \mathcal{C}(A, 1)$ such that (A, π_1, π_2) is a product of A with 1.

2. What is the dual of this statement? That is, what does the statement above concerning $\mathcal{C}$ say about the category $\mathcal{D} = \mathcal{C}^{\mathrm{op}}$?