On some tractable constraints on paths in graphs and in proofs

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Constrained path-finding problems

Problem

Input: undirected graph G, vertices $u, v \in V(G)$, additional data D
Output: a path p between u and v
p must satisfy constraints depending on D

- e.g. for directed graphs, $D =$ edge directions
- Such problems are often either:
  - reducible to undirected reachability (L)
  - reducible to directed reachability (NL)
  - reducible to **alternating paths for matchings**
  - NP-complete
- This talk: focus on problems *equivalent to* alternating paths
  - Note: directed path = alt. path for *bipartite* matching, seems strictly easier than for general matchings

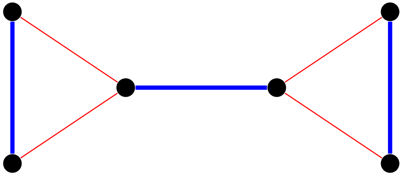
Alternating paths and cycles / Terminology

- Importance of alt. paths: combinatorial matching algorithms

Lemma (Berge 1957)

A matching is maximum *iff it admits no alternating* path.
A perfect matching is unique *iff it admits no alternating* cycle.

- Here, *path* (resp. *cycle*) means without repeating *vertices*
- From now on, *trail* (resp. *closed trail*) means w/o repeating *edges*
- Unconstrained: $\forall u, v$, $\exists$ path $\Leftrightarrow \exists$ trail $\Leftrightarrow \exists$ walk
- Alternating for a matching: $\exists$ path $\Leftrightarrow \exists$ trail $\not\Leftarrow \exists$ walk



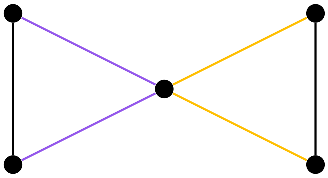
Example: properly colored paths

- Edge-colored graph: equipped with $E \to \{\text{colors}\}$

Definition

A path/trail/walk is *properly colored* (*PC*) if consecutive edges have different colors.

- Generalizes alternating paths, but $\exists$ PC path $\not\Leftarrow \exists$ PC trail



- Conversely, alt. paths can encode PC paths (Szeider 2003, Gutin & Kim 2009) and PC trails (Abouelaoualim et al. 2008)

A family of equivalent problems (1)

- In what sense are these problems *equivalent*?
- One possible meaning: complexity-theoretic reductions

Theorem

Alternating paths for general matchings can be found in linear time.

Corollary

Properly colored paths and trails can be found in linear time.

- But these reductions are not merely algorithmic: they also transfer structural properties
  - Next slide: structural theorems which can all be proved from one another

A family of equivalent problems (2)

Theorem (Kotzig 1959)

Every unique perfect matching (i.e. w/o alt. cycle) contains a bridge.

Theorem (Yeo 1997)

An edge-colored graph without PC cycle has a "color-separating vertex".

Theorem (Abouelaoualim et al. 2008)

An edge-colored graph without PC closed trail has either a vertex with ≤ 1 incident color, or a bridge.

- To sum up: tractable path-finding + "structure from acyclicity"
- Many more constraints belong to this family: Szeider, *On theorems equivalent with Kotzig's result on graphs with unique 1-factors*, 2004

Our results

- We exhibit new members of this family:
  - trails avoiding *forbidden transitions*
  - a special case of *rainbow paths*
- + a dichotomy theorem: other cases of rainbow paths are all NP-complete
- And another equivalent problem coming from logic, whose theory has been independently investigated by logicians
  - They discovered "structure from acyclicity", but not linear-time path-finding

Forbidden transitions

Definition

Let $G = (V, E)$ be a multigraph. A *transition graph* for a vertex $v \in V$ is a graph whose vertices are the edges incident to v. A *transition system* on G is a family $T = (T(v))_{v \in V}$ of transition graphs.
A path (resp. trail) $v_1, e_1, v_2 \ldots, e_{k-1}, v_k$ is said to be *compatible* if for $i = 1, \ldots, k-1$, e_i and e_{i+1} are adjacent in $T(v_{i+1})$.

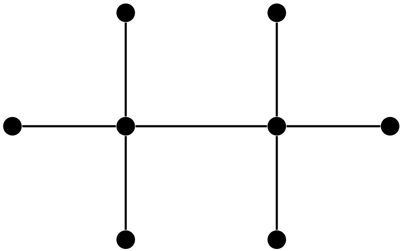
- Very general notion of local constraint
  - Generalizes properly colored paths/trails
- Finding compatible *paths* is NP-complete (Szeider 2003)
- We reduce compatible *trails* to properly colored paths

The edge-colored line graph

Definition

Let $G = (V, E)$ be a multigraph and T be a transition system on G. The *EC-line graph* $L_{EC}(G, T)$ is formed by

- taking the line graph of G,
- coloring its edges according to the vertices of G they come from,
- and deleting the edges corresponding to forbidden transitions.



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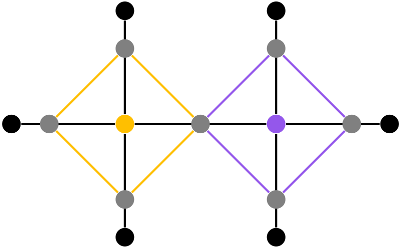


Results on compatible trails

Lemma

Trails in G compatible with T correspond to properly colored paths in $L_{EC}(G, T)$ (bijectively, modulo technical details).

Theorem

Finding a compatible trail can be done with a time complexity linear *in the number of* allowed *transitions (thus, in at most $O(|E|^2)$ time).*

Theorem ("Structure from acyclicity")

If, for all vertices v in G, the transition graph $T(v)$ is connected, *and G has no closed trail compatible with T, then G has a bridge.*

- Generalizes the result on PC trails mentioned earlier

Rainbow paths

- Actually, compatible *paths* can also be read from the EC-line graph

Definition

A path is *properly colored* if *consecutive* edges have different colors.
A path is *rainbow* if *all* its edges have different colors.

Lemma

Paths *in G compatible with T correspond to* rainbow *paths in $L_{EC}(G, T)$ (bijectively, modulo technical details).*

- Corollary: since finding compatible paths is NP-hard, so is finding rainbow paths
  - This was already known (Chakraborty et al. 2011)
  - But we can be more precise

Precise NP-hardness for rainbow paths

- Szeider established a *dichotomy theorem* for compatible paths: if we try to restrict the shape of the transition graphs,
  - either the problem is still NP-hard,
  - or it can be solved in linear time,

  and we have a criterion to know in which case we are.
- Together with an adaptation of the reduction by Chakraborty et al., this allows us to prove this new result:

Theorem

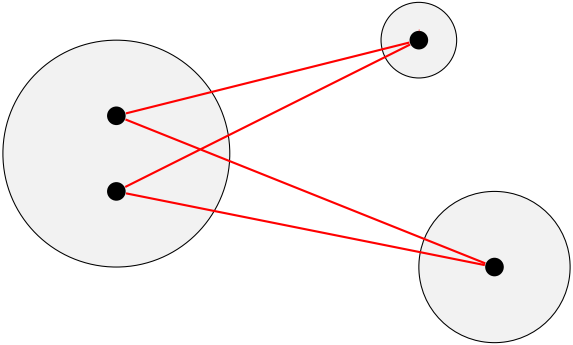
Unless all graphs in a class $\mathcal{A}$ are complete multipartite, *finding a rainbow path in an edge-colored graph whose color classes are in $\mathcal{A}$ is* NP-complete.

- However, if $\mathcal{A}$ is the class of complete multipartite graphs, then it is equivalent to alt. paths / PC paths / etc., and therefore tractable
  - In fact, (non-trivially) linear-time solvable
- Thus, we have a dichotomy theorem for rainbow paths

Structure from rainbow acyclicity

Theorem

If an edge-colored graph whose color classes are complete multipartite has no rainbow cycle, then it contains the kind of configuration described below.



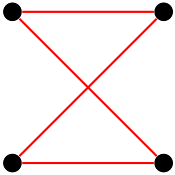
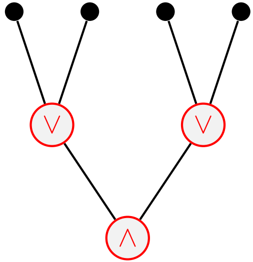
- Next, let's talk about logic

Look, other people actually care about this stuff

- (One variant of) *correctness of proof nets* in *linear logic* belongs to our family of equivalent problems
  - Theory independently reinvented by the linear logic community (J.-Y. Girard, V. Danos, C. Retoré…) starting from the 80's
- Retoré remarked in the mid-90's that proof nets could be translated into graphs equipped with perfect matchings
  - Paper only published in 2003
  - Note: Retoré's PhD thesis studies edge-colored graphs with *bipartite* color classes, and proves "structure from rainbow acyclicity" for them
- Recently, I showed that there actually is an equivalence
  - *Unique perfect matchings and proof nets*, FSCD 2018
  - Also, the EC-line graph construction comes from an analysis of Retoré's reduction

Quick reminder on cographs

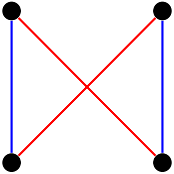
- Cographs are the class of graphs generated by $\vee$ (disjoint union) and $\wedge$ (its dual)
- Can be represented as cotrees (i.e. modular decomposition trees)
  - Cotrees are computable in linear time

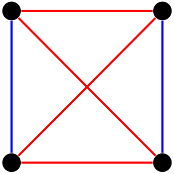



Proof nets for graphs theorists (1): cographs

Definition

A *cographic proof* is an pair of graphs (G, M), G being a cograph and M a 1-regular graph, with the same set of vertices.
A *vicious circle* in (G, M) is a *chordless* cycle in $G \cup M$ which alternates between edges in G and edges in M. A cographic proof is *correct* if it contains no vicious circle.

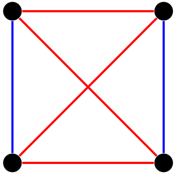


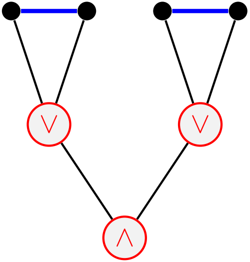


- This presentation of proof nets is due to Retoré
- Correctness related to orthogonality in Sellier's interaction graphs

Proof nets for graphs theorists (2): cotrees

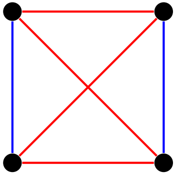
- Take the cotree of G, and represent M as edges between the leaves

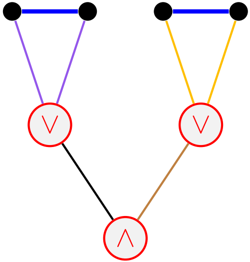




- This is more or less what logicians call proof nets
- Think of the cotree as the syntax tree of a logical formula

Proof nets for graphs theorists (3): edge-colored cotrees





- With this edge coloring, $\exists$ properly colored cycle in proof net (leaf-paired cotree) $\Leftrightarrow \exists$ vicious circle
- $\rightarrow$ correctness $\Leftrightarrow$ constrained acyclicity belonging to our family
- In this case $\exists$ PC cycle $\Leftrightarrow \exists$ PC closed trail

Proof nets for graphs theorists (4): sequentialization

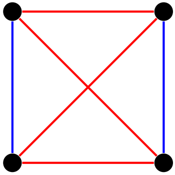
- What does "structure from acyclicity" correspond to in proof net theory?
- It becomes a lemma used to prove an *inductive characterization* of correct proofs
- This reflects the inference rules of linear logic
  - Cographic proofs / proof nets are proofs in which the order of reasoning steps has been forgotten
  - Logicians care about the possible orderings of inference for a given proof
- One can define the "bridge deletion order" of a unique perfect matching, and translate theorems of linear logic into graph theory
  - $\rightarrow$ characterization of this order using *blossoms* (translation of a proof-theoretic result of Bellin 1997)

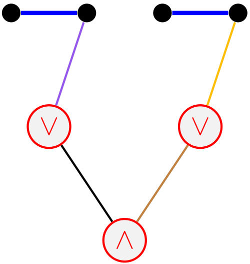
An easier family of problems (1)

Theorem

For a cographic proof, the following are equivalent:

- *it is correct and has a chordless alt. path between any two vertices*
- *all maximal rainbow subgraphs of its leaf-paired cotree are spanning trees*
- *it is inductively generated without the "disjoint union" rule*





An easier family of problems (2)

- This "acyclic-connected" condition is more natural from the POV of logic than merely acyclic
- It can be tested in NL, unlike general alt. cycles (?)
- But is there a simple, purely graph-theoretic equivalent problem, with similar structural properties (e.g. inductive generation)?
  - Looking for "tree-like" instead of "forest-like" conditions
- A candidate: edge-colored graphs whose maximal subgraphs are all spanning trees (cf. previous theorem)
  - Tractable *without any additional conditions*, using "contractibility" from linear logic

Lemma

A "rainbow tree-like" graph is a spanning subgraph of one with bipartite *color classes.*

Conclusion

- New results on natural constrained path-finding problems
  - Equivalence of compatible trails with alternating paths, with "structure from acyclicity" property
  - Dichotomy theorem for rainbow paths
- Connections with proof theory, yielding new theorems and suggesting new questions
  - Not mentioned here: applying graph algorithms to linear logic

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Thank you for your attention!