On some tractable constraints on paths in graphs and in proofs

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Abstract

We show that trails avoiding forbidden transitions and rainbow paths for complete multipartite color classes can be found in linear time, whereas finding rainbow paths is NP-complete for any other restriction on color classes. For the tractable cases, we also state new structural properties equivalent to Kotzig's theorem on bridges in unique perfect matchings. Finally, we mention some connections with proof nets in linear logic and combinatorial proofs ("proofs without syntax") for classical propositional logic.

Keywords : *Perfect matchings, forbidden transitions, properly colored paths, rainbow paths.*

1 Introduction

Many problems which consist of finding a path or trail[1] under some constraints between two given vertices are equivalent to the *augmenting path* problem for matchings, and thus tractable. Some of these problems have associated "structure from acyclicity" theorems which were shown [13] to be equivalent to Kotzig's theorem on the existence of bridges in unique[2] perfect matchings (cf. [13, Theorem 1]): the absence of constrained cycles or closed trails entails the positive existence of some structure in the graph.

Our results here consist of finding new members of this family of constraints on paths which are equivalent in a certain sense, and excluding other constraints through NP-hardness results. We also bring to attention the fact that this family has a representative in proof theory.

Edge-colored graphs From an assignment of colors to the edges of a graph, one can define either *local* or *global* constraints:

- In a *properly colored* (PC) path (see [2, Chapter 16]) or trail (see [1]), *consecutive* edges must have different colors. Both can be found in linear time by reduction to augmenting paths, and conversely augmenting paths are a special case of both these problems. The structural result for PC cycles is Yeo's theorem on cut vertices separating colors [2, §16.3].
- In a *rainbow* (also called *heterochromatic* or *multicolored*) path, *all* edges have different colors. The subject of *rainbow connectivity* has been an active area of research recently, but the problem is NP-complete [4] in the general case.

For rainbow paths, we investigate whether restrictions on the shape of the *color classes* – that is, the subgraphs induced by all edges of a given color – make the problem tractable, and we establish that there is a single case which is not NP-hard:

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[1] Following a common usage (see e.g. [2, Section 1.4]), a *path* is a walk without repeating *vertices* and a *trail* is a walk without repeating *edges*; a *cycle* (resp. *closed trail*) is a closed walk without repeating vertices (resp. edges). Paths (resp. cycles) are trails (resp. closed trails), but the converse does not always hold.

[2] Recall that a perfect matching is unique if and only if it admits no alternating cycle.

Theorem 1. *Let $\mathcal{A}$ be a class of graphs without isolated vertices*[3]*. The rainbow path problem for graphs whose color classes are all in $\mathcal{A}$ can be solved in* linear time *if all graphs in $\mathcal{A}$ are* complete multipartite*, and is* NP-complete *otherwise.*

The first case is part of our family of equivalent constraints, and the associated structural theorem is as follows:

Theorem 2. *Let G be an edge-colored graph whose color classes are complete multipartite. If G has no rainbow cycle, then there exists a color c such that for all c-colored edges (u, v), u and v are in different connected components after removing the color class of c.*

Forbidden transitions A very general notion of *local* constraints is to simply forbid some pairs of edges from occuring consecutively in a path. We take the following definition from [12].

Definition 1. Let $G = (V, E)$ be a multigraph. A *transition graph* for a vertex $v \in V$ is a graph whose vertices are the edges incident to v. A *transition system* on G is a family $T = (T(v))_{v \in V}$ of transition graphs.

A path (resp. trail) $v_1, e_1, v_2 \dots, e_{k-1}, v_k$ is said to be *compatible* (or *avoiding forbidden transitions*) if for $i = 1, \dots, k-1$[4], e_i and e_{i+1} are adjacent in $T(v_{i+1})$.

That is, the edges of the transition graphs specify the *allowed* transitions. Finding a compatible *path* has been proven to be NP-complete [12]. However, the question for compatible *trails* does not seem to have been asked before in its full generality. We show that:

Theorem 3. *Finding a compatible trail can be done with a time complexity* linear *in the number of* allowed *transitions (thus, in at most quadratic time in the size of the graph).*

Theorem 4 ("Structure from acyclicity")**.** *Let G be a multigraph with transition system T, with at least one edge. If, for all vertices v in G, the transition graph $T(v)$ is* connected*, and G has no closed trail compatible with T, then G has a bridge.*

Corollary 1 (New[5] proof of [1, Theorem 2.4])**.** *Let G be an edge-colored graph such that every vertex of G is incident with at least two differently colored edges. Then, if G does not have a PC closed trail, then G has a bridge.*

2 The edge-colored line graph

A key ingredient in the aforementioned results is a kind of *line graph* construction mapping graphs with forbidden transitions to edge-colored graphs.

Definition 2. Let $G = (V, E)$ be a multigraph and T be a transition system on G. The *EC-line graph* $L_{EC}(G, T)$ is formed by taking the line graph of G, coloring its edges so that the clique corresponding to v is given the color v (using the vertices of G as the set of colors), and deleting the edges corresponding to forbidden transitions.

Formally, $L_{EC}(G, T)$ is defined as the graph with vertex set E and edge set $F = \bigsqcup_{v \in V} T(v)$, equipped with an edge coloring $c : F \to V$ with values in V: for $f \in F$, $c(f)$ is the unique vertex such that $f \in T(c(f))$.

Proposition 1. *Let G be a multigraph with transition system T, and $s \neq t$ be vertices of G.*

The compatible paths *between s and t correspond bijectively to* rainbow paths *in $L_{EC}(G, T)$ between some vertex of $\partial(s)$ and some vertex of $\partial(t)$ which do not cross edges with color s or t.*

Similarly, the compatible trails *between s and t where neither s nor t appear as intermediate vertices correspond bijectively to* PC paths *in $L_{EC}(G, T)$ between some vertex of $\partial(s)$ and some vertex of $\partial(t)$ which do not cross any edge with color s or t.*

[3]Indeed, a color class, which is an edge-induced graph, cannot have isolated vertices.

[4]For a cycle (resp. closed trail), we must also require e_{k-1} and e_1 to be adjacent in $T(v_1) = T(v_k)$.

[5]The original proof applies Yeo's theorem to a construction which does not generalize to forbidden transitions, but provides a trail-finding algorithm in linear time *in the size of the graph*.

Theorems 3 and 4 immediately follow from the second half of this proposition together with the known results on PC paths. However, to get the hardness result for rainbow paths, in addition to the EC-line graph, we need to reuse the proof techniques from [12] and [4], in particular a characterization of complete multipartite graphs by excluded vertex-induced subgraphs [12, Lemma 7]. As for the first half of Theorem 1, it uses the fact that one can retrieve the vertex partition of a complete multipartite graph in linear time, for instance by computing its cotree [5].

3 Constrained cycles in logic

In a recent work [9], we showed that the *correctness* of a *proof net* – a graph-like representation of a proof in *linear logic* [6] – is equivalent to the uniqueness of a given perfect matching, and is therefore part of our family of equivalent problems. Thus, it can be decided in linear time, and the associated structural property is the key lemma in the proof of the "sequentialization theorem", an inductive characterization of the set of correct proof nets which mirrors exactly the inference rules of linear logic.

One direction of the equivalence, from proof nets to perfect matchings, had been established previously by Retoré [11, §1][6]. His reduction can be understood *a posteriori* as a composition of constructions on edge-colored graphs: it amounts to equipping a proof net with a transition system, taking the EC-line graph introduced above, and applying a known reduction from edge-colored graphs with chromatic degree ≤ 2 to perfect matchings [8][7].

Let us give a rough presentation of proof nets in graph-theoretic terms. A proof net may be seen as the syntax tree of a propositional formula, with $\wedge$ and $\vee$ nodes and literals at the leaves, together with additional edges between the leaves pairing together opposite literals. The syntax tree may be interpreted as the *cotree* of a *cograph* whose vertices are the literals, as usual, see e.g. [3]. This leads to a restatement of correctness, also due to Retoré [11, §2].

Definition 3. A *cographic proof* is an pair of graphs (G, M), G being a cograph and M a 1-regular graph, with the same set of vertices.

A *vicious circle* in (G, M) is a *chordless* cycle in[8] $G \cup M$ which alternates between edges in G and edges in M. A cographic proof is *correct* if it contains no vicious circle.

A proof net is correct if and only if the corresponding cographic proof (with the 1-regular graph representing the pairing of the leaves) is correct in the sense above. Note that vicious circles are not merely properly colored cycles for the natural 2-edge-coloring of the cographic proof, because of the additional chordlessness condition.

Finally, let us mention that cographic proofs also have applications outside of linear logic. Indeed, they have been used to define "proofs without syntax" for classical propositional logic: Hughes's *combinatorial proofs* [7] are graph homomorphisms (with additional properties) from some correct cographic proof to the cograph of the classical formula being proven, and this gives a sound and complete proof system. The tractability of our family of constraints on cycles ensures that proofs are checkable in polynomial time.

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[6] This was the first indication of a connection between linear logic and unique perfect matchings. Let us mention as well that in an earlier attempt to connect linear logic with graph theory [10, Chapter 2], Retoré proved a weaker version of the structural theorem for rainbow acyclic graphs (it requires the color classes to be complete *bipartite* instead of complete multipartite).

[7] This paper only defines the reduction for 2-edge-colored graphs, but the required generalization is straightforward. Note also that the two last steps give a direct reduction from compatible trails to perfect matchings. Although we have not managed to find it in the literature, there is at least one other place where it occurs implicitly, which also inspired us: a solution to an algorithmic puzzle by Christoph Dürr, see `http://tryalgo.org/en/matching/2016/07/16/mirror-maze/`.

[8] By $G \cup M$ we mean the graph whose edges are the union of those in G and M, on the common vertex set. This union may result in a multigraph with parallel edges.

4 Conclusion and perspectives

We summarize the complexity of the problems studied here in the following table. Our contributions, marked in bold, fill some gaps in the table, thus answering several natural questions. Furthermore, we exhibited a construction which provides a bridge between different kinds of constraints on paths and trails, and described how different reductions relate to each other.

	Time complexity / additional results
Path avoiding forbidden transitions	NP-complete with dichotomy result [12]
Trail avoiding forbidden transitions	**Linear with structural theorem**
Properly colored path	Linear with structural theorem (cf. [2])
Properly colored trail	Linear with structural theorem [1]
Rainbow path/trail[9] (general)	NP-complete [4], **with dichotomy result**
Rainbow path/trail (restricted[10])	**Linear with structural theorem**

To clarify, the connection with proof nets works specifically for a system called Multiplicative Linear Logic with the Mix rule. Without this Mix rule, correctness becomes a "tree-like" condition instead of an acyclicity ("forest-like") condition.

What analogous conditions could one ask of a constrained graph? In the case of rainbow paths and cycles, we may consider edge-colored graphs whose maximum rainbow subgraphs are all trees. Remarkably, it seems that we have a polynomial-time recognition algorithm and a structural property for these graphs without any restriction on the shape of color classes[11].

References

[1] A. Abouelaoualim, K. Ch. Das, L. Faria, Y. Manoussakis, C. Martinhon, and R. Saad. Paths and trails in edge-colored graphs. *Theoretical Computer Science*, 409(3):497–510, December 2008.

[2] Jørgen Bang-Jensen and Gregory Gutin. *Digraphs. Theory, algorithms and applications.* 2nd ed.

[3] Seth Chaiken, Neil V. Murray, and Erik Rosenthal. An application of P_4-free graphs in theorem-proving. *Annals of the New York Academy of Sciences*, 555(1):106–121, May 1989.

[4] Sourav Chakraborty, Eldar Fischer, Arie Matsliah, and Raphael Yuster. Hardness and algorithms for rainbow connection. *Journal of Combinatorial Optimization*, 21(3):330–347, April 2011.

[5] Derek G. Corneil, Yehoshua Perl, Lorna K. Stewart. A linear recognition algorithm for cographs. *SIAM Journal on Computing*, 14(4):926–934, 1985.

[6] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–101, January 1987.

[7] Dominic J.D. Hughes. Proofs without syntax. *Annals of Mathematics*, 143(3):1065–1076, 2006.

[8] Yannis Manoussakis. Alternating paths in edge-colored complete graphs. *Discrete Applied Mathematics*, 56(2):297–309, January 1995.

[9] Lê Thành Dũng Nguyễn. Unique perfect matchings and proof nets. In *3rd International Conference on Formal Structures for Computation and Deduction, FSCD 2018*, 26:1–26:19.

[10] Christian Retoré. *Réseaux et séquents ordonnés.* PhD thesis, Université Paris VII, February 1993.

[11] Christian Retoré. Handsome proof-nets: perfect matchings and cographs. *Theoretical Computer Science*, 294(3):473–488, February 2003.

[12] Stefan Szeider. Finding paths in graphs avoiding forbidden transitions. *Discrete Applied Mathematics*, 126(2-3):261–273, 2003.

[13] Stefan Szeider. On theorems equivalent with Kotzig's result on graphs with unique 1-factors. *Ars Combinatoria*, 73, 2004.

[9] The existence of a rainbow path is equivalent to the existence of a rainbow trail between two vertices.

[10] Restricted to edge-colored graphs with complete multipartite color classes.

[11] The trick is that any such graph is a spanning subgraph of another with complete bipartite color classes.