On the complexity of finding cycles in proof nets

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Abstract

By importing results from graph theory, in particular the theory of edge-colored graphs, we show that correctness of proof nets with the Mix rule can be decided in linear time, and characterize its sub-polynomial complexity. On the other hand, we establish the coNP-hardness of Pagani's visible acyclicity condition on MELL proof structures.

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1 Introduction

This work is about revisiting the *proof nets* of linear logic in the light of graph theory; in particular, we study the complexity of decision problems on proof nets by taking advantage of the vast literature on graph algorithms. Recently, building on previous work by Retoré [12], we established a connection [9] between proof nets for Multiplicative Linear Logic with the Mix rule (MLL+Mix) and *unique perfect matchings*. (A perfect matching is a subset of edges in a graph such that every vertex is incident to exactly one edge in this set.)

This allowed us to make progress on the problem of *correctness* for MLL+Mix – that is, distinguishing proof nets from similar graphs which do not correspond to valid proofs, called *proof structures*. Even though a lot of work has been devoted to proof net correctness, it has been mainly focused on MLL without Mix, and has generally made use of ad hoc structures, e.g. *paired graphs*. Regarding the complexity of correctness, our connection manifests as follows:

▶ **Theorem 1.** *The following problems are equivalent under linear-time and* AC^0 *reductions:*
- *correctness of proof structures for MLL+Mix;*
- *deciding, given a perfect matching in a graph, whether it is unique.*

In fact, this extends without difficulty to Multiplicative Exponential Linear Logic (MELL) with Mix and quantifiers. Since perfect matchings are a central object of study in mainstream graph theory, we can use previously known results on matchings to draw immediate consequences from the equivalence above:
- correctness can be decided in linear time (this is new for MLL+Mix, but for MLL without Mix, this was previously shown by Guerrini [5]), in *randomized* NC and in quasi-NC (the class decided by polylogarithmic depth and quasipolynomial size circuits);
- a NC correctness criterion would give a NC decision procedure for the existence of a unique perfect matching in a graph, this would in fact settle a 40 year old fundamental conjecture in graph theory [7] (however, for MLL without Mix, correctness is known to be in NC – in fact, it is NL-complete [6]).

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In our previous work, to prove this equivalence, we gave a direct reduction from proof structures to perfect matchings. Here, we show how this direction of the equivalence can be obtained in an indirect fashion, through *edge-colored graphs* [1, Chapter 16] which generalize the usual paired graphs. Extending this, we reduce a coNP-complete problem on edge-colored *directed* graphs to Pagani's *visible acyclicity* [10, 11], a sort of "correctness criterion" on MELL proof structures which is weaker than MELL+Mix correctness, but still guarantees strong normalization. This entails the coNP-hardness of visible acyclicity, which answers in the negative a question[1] by Pagani.

Note that acyclicity of edge-colored graphs is already known to be equivalent to the uniqueness of perfect matchings [13] both from the point of view of decision procedures and of structural properties. But our presentation here does not amount to exactly the same thing as our direct encoding of proof structures into perfect matchings: in the latter, we encode the direction of the edges from premise to conclusion, in a way that meshes well with the structure of unique perfect matchings (and we have a *quasi-linear sequentialization algorithm* which crucially relies on this) whereas this information is lost in our translation to edge-colored graphs, just like in paired graphs (Retoré's previous reduction [12] from proof structures to perfect matchings also had this drawback).

2 From paired graphs to edge-colored graphs

It is well-known that the correctness of a MLL+Mix proof net π with n-ary axiom links (used to handle MELL boxes) and quantifiers reduces to the acyclicity of the switchings of the correctness graph of π (this is a variant of the Danos–Regnier criterion for MLL). This correctness graph is a paired graph.

▶ **Definition 2.** An *edge-colored graph* is an undirected graph $G = (V, E)$ equipped with a mapping from the edges to a finite set of colors. A *rainbow subgraph* of G is a subgraph of G whose edges all have different colors.

A *paired graph* is an edge-colored graph such that for every color, there exists a common endpoint for all edges of this color.

While this is not the most conventional definition of paired graphs, it is equivalent to the usual one – provided that we admit "pairs" of any cardinality (this is necessary to handle the jumps for the quantifiers anyway), cardinality 1 corresponding to unpaired edges. Switchings can be defined as maximum rainbow subgraphs, and this makes sense for arbitrary edge-colored graphs. A *rainbow cycle* is the same thing as a cycle in some switching.

Unfortunately, finding rainbow cycles is NP-hard [2]. We now explain why it is much more tractable when restricted to paired graphs.

▶ **Definition 3.** A *properly colored cycle* is a cycle *without vertex repetitions* in which two *consecutive* edges always have different colors.

▶ **Proposition 4.** *In a paired graph, properly colored cycles are the same as rainbow cycles without vertex repetitions.*

▶ Remark. Any rainbow cycle contains a rainbow cycle without vertex repetitions.

[1] "It might be interesting knowing whether there is a linear algorithm deciding the visible acyclicity of differential nets [...]." [11]

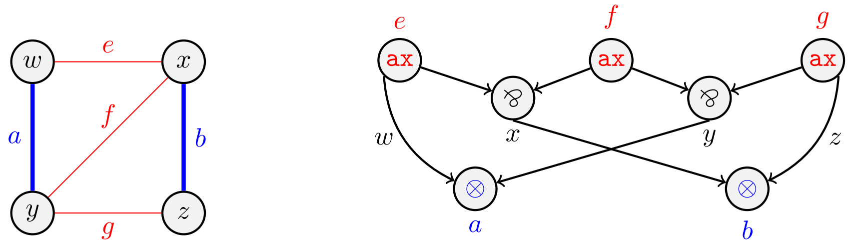

Figure 1 A graph equipped with a perfect matching and its translation into a proof structure.

▶ **Theorem 5** ([1, §16.4]). *The existence of a properly colored cycle in an edge-colored graph reduces to the uniqueness of a given perfect matching in a graph. Furthermore, the reduction can be computed in linear time and in* AC^0.

This proves one direction of the equivalence with uniqueness of perfect matchings.

▶ Remark. There is a third notion of constrained cycle in edge-colored graphs, namely *properly colored closed trails*, which may visit the same vertex multiple times, but cannot cross an *edge* more than once. In the correctness graph of a proof structures, a PC closed trail is the same as a PC path, because for each vertex, there is at most one color such that multiple edges of this color are incident to this vertex.

We see that proof structures collapse different notions of constrained cycles. Distinguishing between them is useful, for instance, to understand the combinatorial content of Retoré's R&B-graph criterion [12]: when generalized to paired graphs, it actually tests for the existence of properly colored closed trails.

3 Encoding 2-edge-colored graphs into proof structures

The other direction of the equivalence is established through a direct reduction from graphs equipped with perfect matchings to proof structures, so that the perfect matching is unique if and only if the proof structure is correct for MLL+Mix. We refer to our previous paper [9] for its formal definition; in short, the idea is to represent edges by $\otimes$-links if they are in the matching, and by `ax`-links if they are outside the matching. An example is provided by Figure 1, with the matching edges in blue and the non-matching edges in red.

In fact, every subset of edges in a graph, and in particular every perfect matching, defines a *2-edge-coloring* of the graph, with the two colors being "in the set" and "outside the set". With this coloring, a perfect matching is unique if and only if there is no properly colored cycle (this is a corollary of *Berge's lemma*).

To test the absence of properly colored cycles in 2-edge-colored graphs, we may use the previously mentioned reduction to perfect matchings, which works for any number of colors, but there is a simpler reduction (attributed to Edmonds in [8]). Basically, make two copies the graph, each having all the original vertices but keeping only the edges of a single color, and join the corresponding vertices with matching edges; see Figure 2 for an example.

By composing these two reductions, we can reduce acyclicity of 2-edge-colored graphs to MLL+Mix correctness, sending vertices to $\otimes$-links and edges to `ax`-links. In the next section, this construction will be adapted to directed graphs.

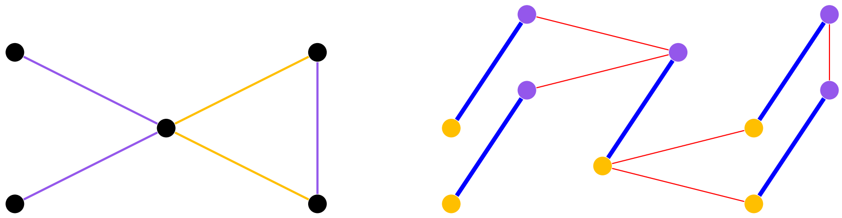

Figure 2 A 2-edge-colored graph and its translation into a graph equipped with a perfect matching

4 Visible acyclicity and edge-colored directed graphs

The *visible acyclicity* condition was first introduced by Pagani for MELL proof structures [10] and later extended to differential interaction nets [11]. It is motivated by semantics[2] and also behaves well under cut-elimination: it is stable under reduction and guarantees strong normalization. Thus, it can be seen as a sort of correctness criterion relaxing the usual MELL+Mix correctness.

The MLL+Mix correctness of a proof structure is characterized by the acyclicity of a family of graphs, namely the switchings of its correctness graphs. Analogously, the visible acyclicity of a MELL proof structure can be defined as the acyclicity of its *visible graphs*. A novelty of visible graphs with respect to switchings is that the former are *directed* graphs. This suggests connections with edge-colored directed graphs. The complexity of the acyclicity problem in this setting is known:

▶ **Theorem 6** ([4]). *Deciding whether a 2-edge-colored directed graph with no properly colored cycle contains a properly colored path between two given vertices is* NP*-complete.*

▶ **Corollary 7.** *Deciding whether a 2-edge-colored directed graph contains a properly colored cycle is* NP*-complete.*

Let us now sketch a reduction from the above problem to the existence of cycles in visible graphs. Recall that for 2-edge-colored undirected graphs, we defined in the previous section a reduction to proof structures, sending edges to `ax`-links. It suffices to replace these links by a gadget realizing a "directed `ax`-link" in order to handle directed edges. An exponential box with a single auxiliary door will do the trick, as long as the box contains a MLL+Mix proof net with no path between the premise of the principal door and that of the auxiliary door. (We will not detail further here the definition of visible acyclicity.)

In the end, we have:

▶ **Theorem 8.** *Visible acyclicity is* coNP*-hard.*

However, it is not clear whether we can express the visible acyclicity of a general MELL proof structure as the absence of properly colored directed cycles in a graph without a superpolynomial size increase. More generally, we do not know whether visible acyclicity is in coNP.

[2] Visible acyclic proof structures (resp. differential interaction nets) are characterized by having a sound denotation in non-uniform coherence spaces (resp. finiteness spaces).

5 Conclusion

We have seen that preexisting results in graph theory, in particular those concerning edge-colored graphs, can help us determine the complexity of decision problems on proof nets. (Our work using a direct translation to perfect matchings [9] also yields efficient algorithms for some function problems.) In particular, the sub-polynomial complexity of MLL+Mix correctness is equivalent to that of a central problem in graph theory.

We mentioned in the introduction that after translating proof structures to edge-colored graphs, we cannot distinguish premises from conclusions anymore. Is there a convincing way to compensate for this loss of information within the edge-colored setting? One could simply add the missing data on top of the edge-colored graph; the resulting object would be equivalent to the "directed graphs with ports" used to formalize proof nets with additional jumps [3]. We believe that it should admit a graph-theoretic sequentialization theorem, generalizing MLL+Mix sequentialization and admitting a quasi-linear algorithm for reconstructing an inductive derivation.

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