A complexity gap between pomset logic and system BV, via perfect matchings in digraphs

Or: proof nets according to Christian Retoré

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joint work with Lutz Straßburger (Inria Saclay)

Journée d'étude pour les 60 ans de Christian Retoré, 15 janvier 2024

What is this about?

Pomset Logic (PL) and *system BV*: 2 logics over the same formulas

A two-decades-old conjecture

These logics are equivalent, i.e. prove the same formulas.

It was known that $(\mathrm{BV} \vdash A) \implies (\mathrm{PL} \vdash A)$.

Our result: refuting the conjecture

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$$A = ((a \lhd b) \otimes (c \lhd d)) \parr ((e \lhd f) \otimes (g \lhd h)) \parr (a^\perp \lhd h^\perp) \parr (e^\perp \lhd b^\perp) \parr (g^\perp \lhd d^\perp) \parr (c^\perp \lhd f^\perp)$$

The classical sequent calculus LK

An usual proof system for classical logic:

- Identity and cut rules
- Logical rules:

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Remove contraction and weakening $\rightarrow$ *Multiplicative Linear Logic* (MLL)

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$$A, B ::= a \mid a^{\perp} \mid A \otimes B \mid A \parr B$$

Involutive negation *defined* by De Morgan rules:

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$\longrightarrow$ semantics may suggest extensions to the logic

Extensions to Multiplicative Linear Logic

The denotational semantics of MLL in (hyper)coherence spaces suggest:

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Two conservative extensions of MLL+Mix with $A, B ::= \cdots \mid A \lhd B$

- Pomset Logic (Christian Retoré, early 1990s) — based on *proof nets*
- System BV (Alessio Guglielmi, late 1990s... also anticipated by Retoré!)
  — 1st application of *deep inference*

Motivations

Guglielmi 2007, *A System of Interaction and Structure* (emphasis mine):

> **It is still open whether the logic in this paper, called *BV*, is the same as pomset logic. We conjecture that it is actually the same logic**, *but one crucial step is still missing, at the time of this writing, in the equivalence proof. This paper is the first in a planned series of 3 papers dedicated to BV.* [...] *In the 3rd part, some of my colleagues will hopefully show the equivalence of BV and pomset logic*

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∃ formal arguments showing that "traditional sequent calculi cannot express BV"
[Tiu 2006]

A glance at deep inference

A methodology originally introduced for BV;

many other successes in past 2 decades (e.g. cut-free proofs for modal logics)

Deep inference = unary rules applied to subformulas of arbitrary depth:

$$\text{inference rule } \frac{A}{B} \qquad \rightsquigarrow \qquad \text{instances } \frac{S[A]}{S[B]} \text{ for any context } S$$

e.g. $$\frac{A \parr (B \otimes (C \parr D))}{A \parr ((B \otimes C) \parr D)}$$

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(compare with rewriting systems, or functoriality in categorical logic)

Deep inference for MLL+Mix and BV

- Identity rules: $\overline{\mathbf{I}}$ and $\dfrac{\mathbf{I}}{a \parr a^\perp}$
- MLL+Mix: rules for assoc/comm. of $\otimes$, $\parr$ + unitality ($A \otimes \mathbf{I} \equiv A \parr \mathbf{I} \equiv A$) +

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- unit-less SBV rules (except identity/cut) first proposed by Retoré!

  Pomset logic as a calculus of directed cographs, 1999 (Inria Research Report)

Proof nets for Multiplicative Linear Logic

The proof system for Pomset Logic extends the graphical syntax of MLL *proof nets*

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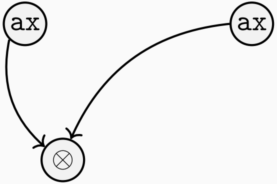

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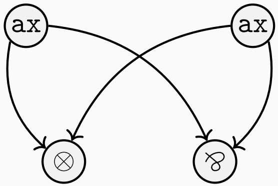

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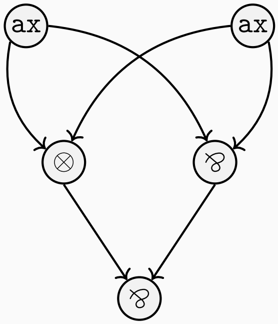

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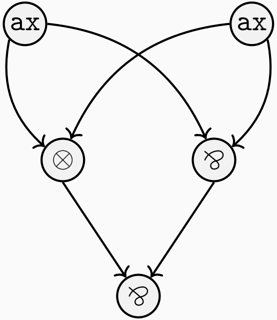


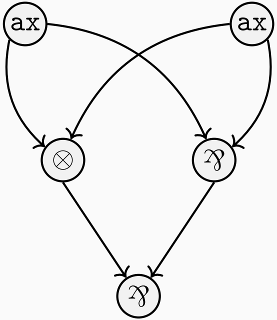
No distinction between $\otimes$ and $\parr$ $\longrightarrow$ not all graphs correspond to correct proofs

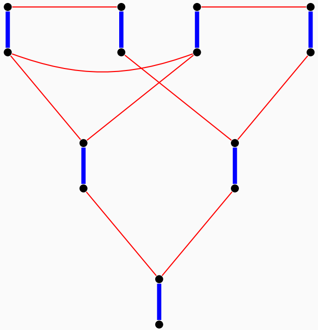
$\longrightarrow$ need a *correctness criterion*

In addition to Pomset Logic, Retoré also invented in the 1990s...

A translation MLL+Mix proof nets $\rightarrow$ graphs equipped with *perfect matchings*

(for linear logicians: reformulation of Danos–Regnier switching criterion)





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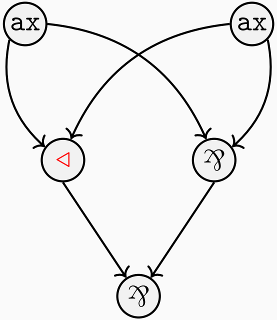

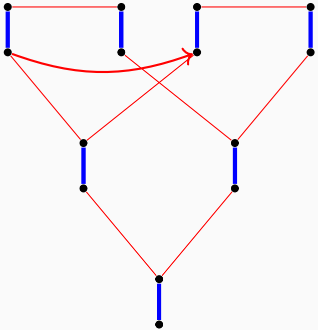

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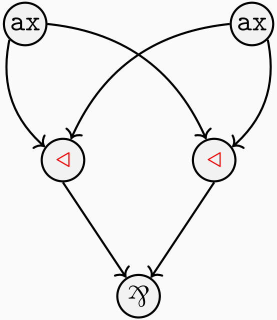

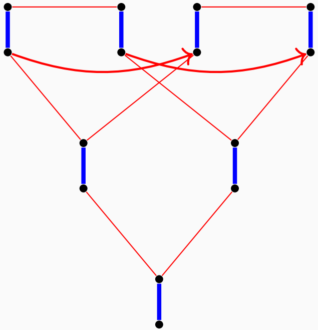

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Complexity of proof net correctness

In previous work (*Log. Methods Comput. Sci.* 2020) I brought proof nets and perfect matchings even closer by exhibiting a converse translation ($\leftarrow$)

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This led us to a surprising realization:

Theorem (N. & Straßburger – LMCS paper, published last month)

Provability in pomset logic is strictly harder *than in BV unless* $\mathsf{NP} = \mathsf{coNP}$.

(*more precisely:* Σ_2^{p}*-complete vs* NP*-complete*)

- In BV, the length of proofs is polynomially bounded
- It's known that finding constrained cycles in directed graphs is often hard

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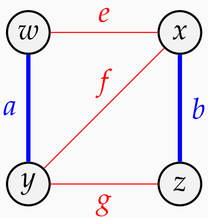
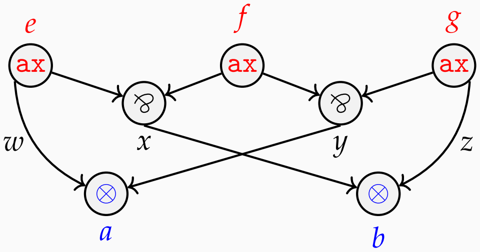
$\longrightarrow$ Suddenly, Guglielmi's conjecture looked less plausible…

Reduction perfect matchings → proof structures (FSCD 2018 / LMCS 2020)





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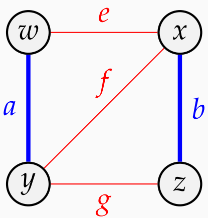
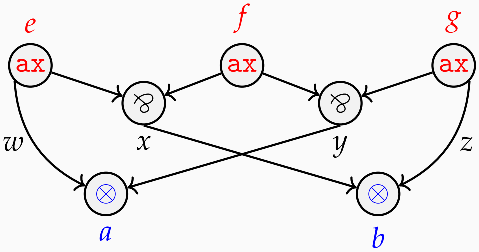




Extends to perfect matchings in general *directed* graphs → *pomset logic* proof structures, using the non-commutative $\triangleleft$ to build a "directed ax" gadget. Hence:

Lemma

There is a PTime reduction: ∃ directed æ-cycle ⇝ pomset logic proof net incorrectness.

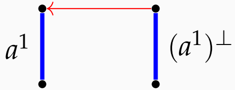
Directed axioms and causality

Aleks Kissinger & Will Simmons (Oxford) consider categories modelling higher-order causal processes (obtained via double-gluing). They show that the "logic of these categories" is precisely a conservative extension of pomset logic.

(*An exact logic for compatibility of higher-order causal structures*, in preparation)

$$A, B ::= a^1 \mid (a^1)^\perp \mid a \mid a^\perp \mid A \otimes B \mid A \parr B \mid A \lhd B$$

Special atoms a^1 represent 1st-order interfaces to processes, with directed axioms:



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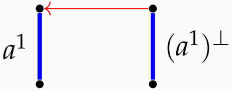

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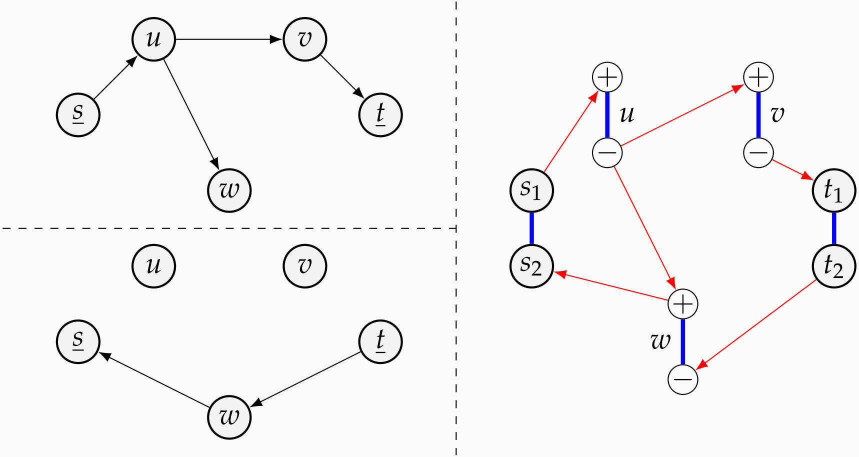
Next goal: show that finding directed æ-cycles is NP-hard

Reduction from another graph-theoretic problem ("elementary round-trip")



cycle $s \xrightarrow{\text{top}} t \xrightarrow{\text{bottom}} s$ *without repeating vertex* $\leftrightsquigarrow$ æ-cycle

Hardness results

The "elementary round-trip" problem is NP-complete ($\overbrace{\text{by reduction from 3SAT}}^{\text{backup slide}}$), so:

Theorem

Pomset proof net correctness is coNP*-complete.*

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Theorem

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We can reduce a Π_2^p-complete variant of elementary round-trip involving the "switchings" of two "paired graphs" [*à la* Danos–Regnier] to pomset *non-provability*, therefore:

Theorem

Pomset logic provability is Σ_2^p-complete.

Remark: here paired graphs / switchings are *not* related to the correctness criterion but to the choice of plugging of axiom links

Conclusion

Retoré's *Pomset Logic* (PL) and Guglielmi's *BV*: 2 logics over the same formulas, from the 1990s, conservatively extending Multiplicative Linear Logic with Mix

Our result [N. & Straßburger]: refuting Guglielmi's two-decades-old conjecture

- *There is some formula A such that* $\mathrm{BV} \nvdash A$ *but* $\mathrm{PL} \vdash A$.

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Causally meaningful variant (K.–S.): $(((p^1)^\perp \lhd q^1) \otimes ((r^1)^\perp \lhd s^1)) \parr (((q^1)^\perp \lhd r^1) \otimes ((s^1)^\perp \lhd p^1))$

- Moreover, "$\mathrm{BV} \vdash A$?" is NP-complete while "$\mathrm{PL} \vdash A$?" is Σ_2^{p}-complete.

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Joyeux anniversaire Christian !

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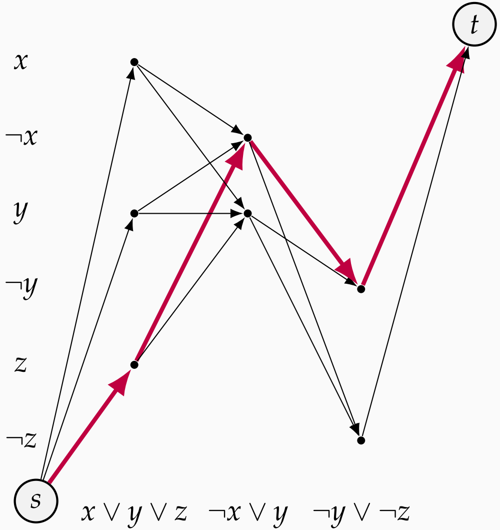

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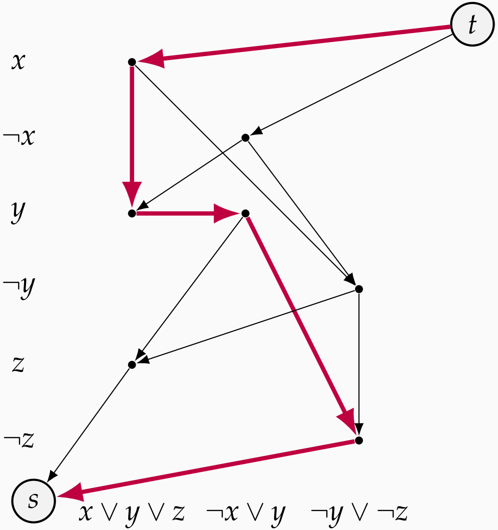

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Non-intersecting pair = satisfying assignment