Revisiting the growth of polyregular functions

Lê Thành Dũng (Tito) Nguyễn — nltd@nguyentito.eu — ÉNS Lyon
joint work with Sandra Kiefer (Oxford) & Cécilia Pradic (Swansea)

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What is an interesting class of finite-state computable functions?

Regular languages ($L \subseteq \Sigma^*$): a robust notion

deterministic finite automata $\iff$ nondeterministic FA $\iff$ two-way FA $\iff$ regular expressions $\iff$ monadic second-order logic (MSO) $\iff$ ...

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- Linear growth: $|f(w)| = O(|w|)$ for $f\colon \Sigma^* \to \Gamma^*$ rational (NFT) / regular (MSO)
- Or hyperexponential growth (L-systems, iterated pushdown transducers, ...)

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Complexity theory: feasible = P. What is the finite-state counterpart?

Proposal (Bojańczyk 2018): polyregular functions

Robust class of string functions, computed by *pebble transducers* (early 2000s)

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by *k*-pebble transducers ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

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$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

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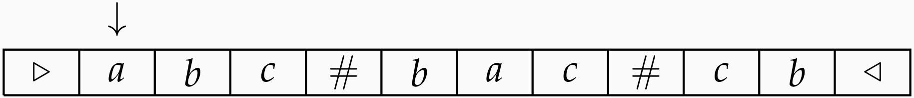

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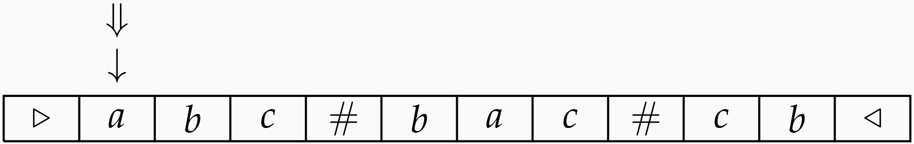

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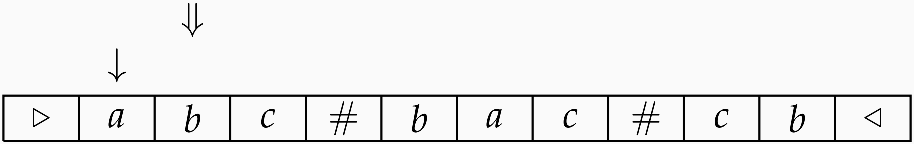

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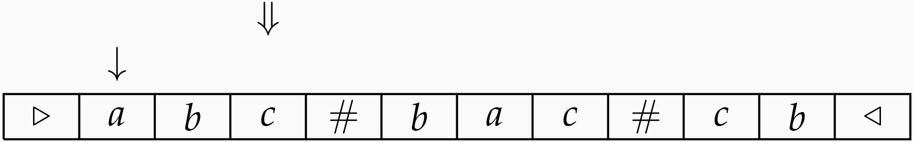

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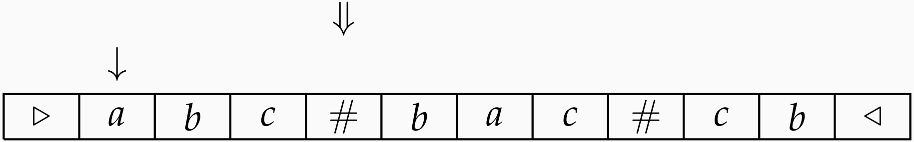

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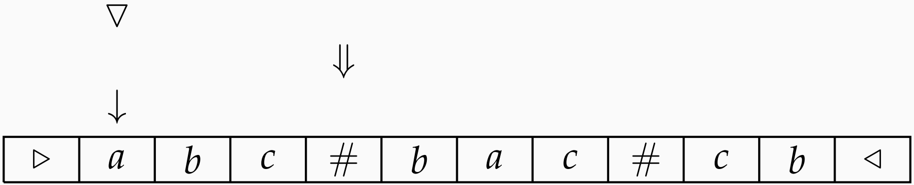

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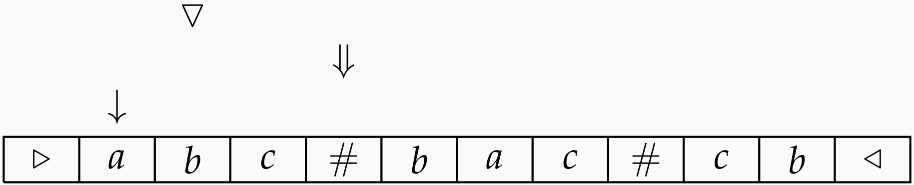

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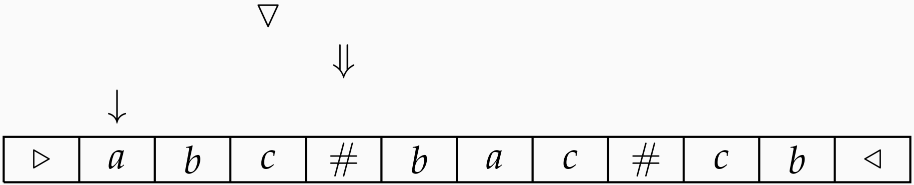

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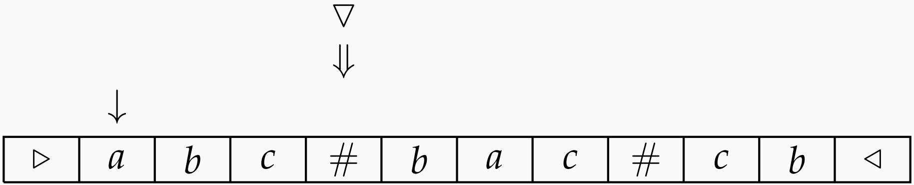

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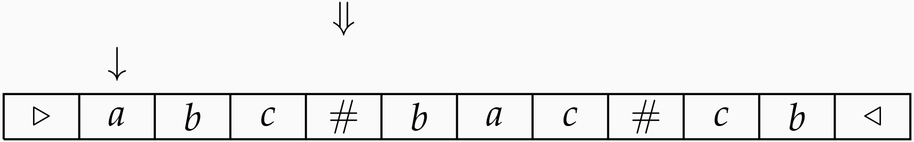

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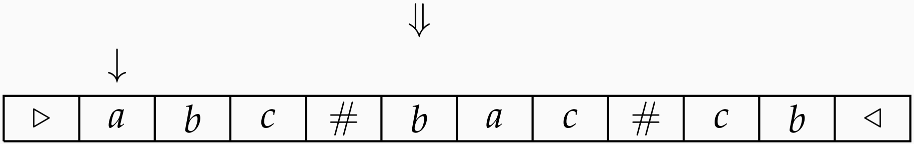

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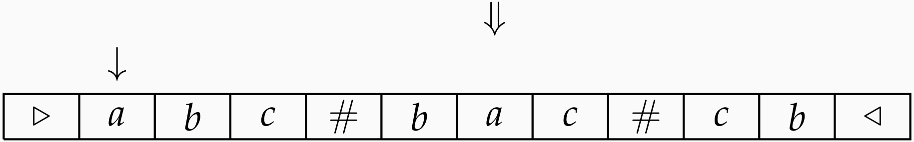

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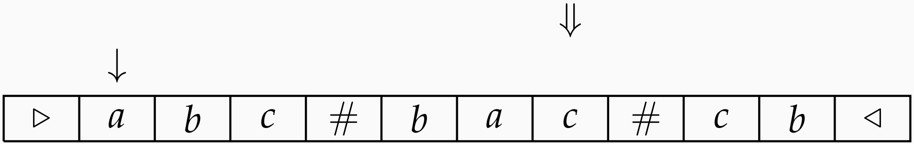

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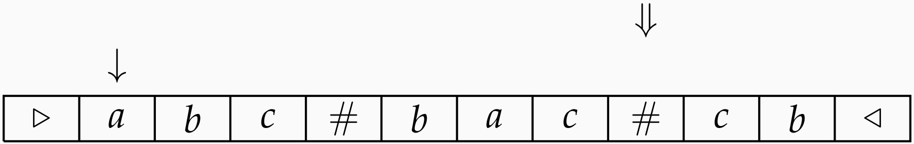

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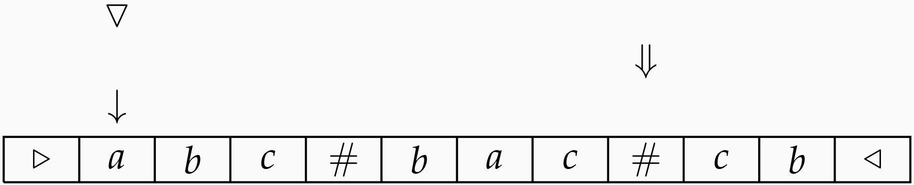

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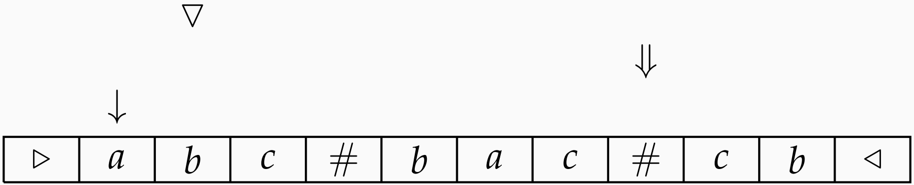

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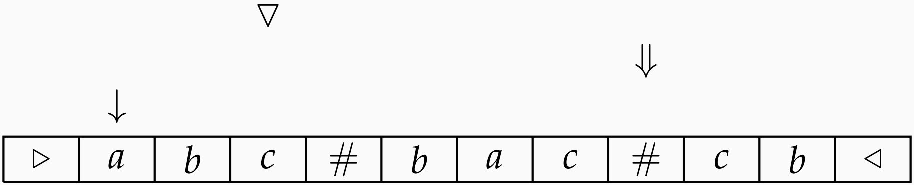

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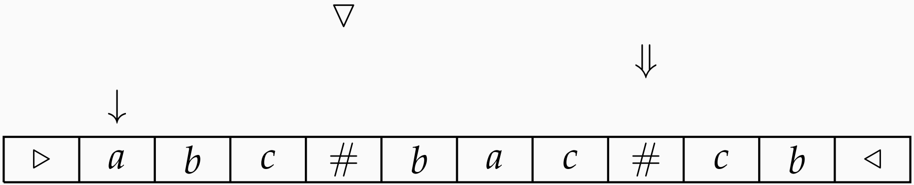

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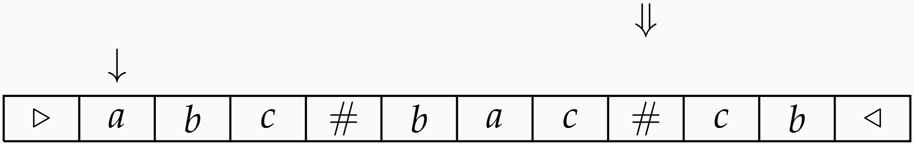

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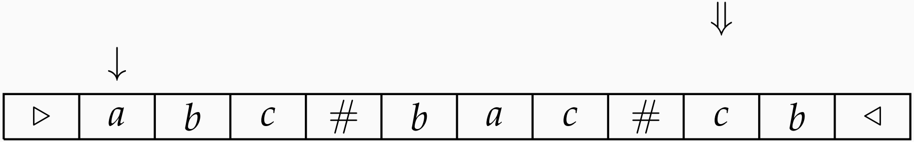

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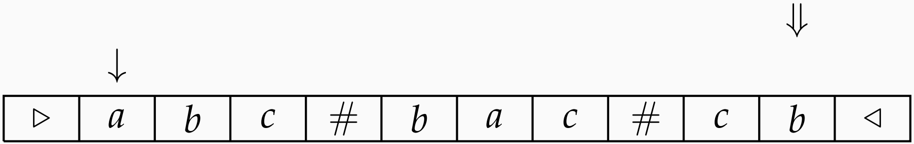

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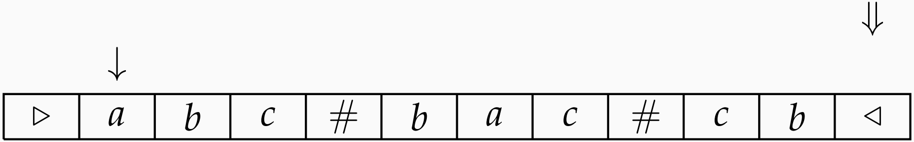

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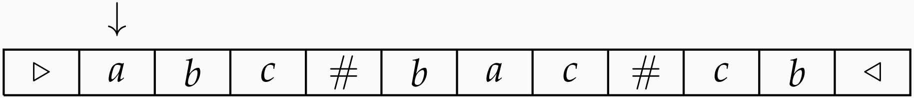

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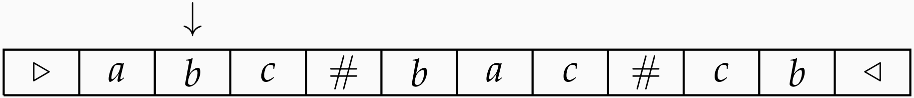

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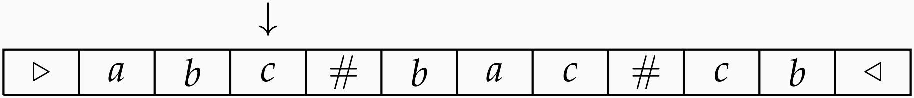

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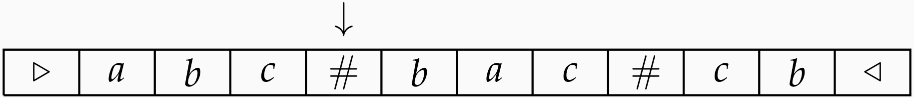

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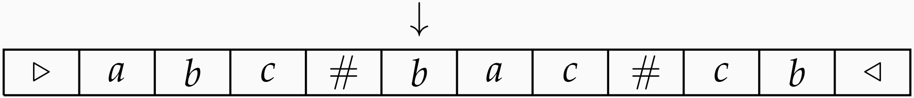

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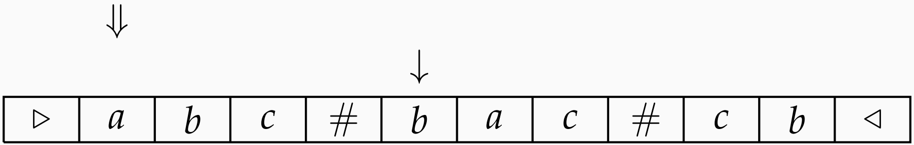

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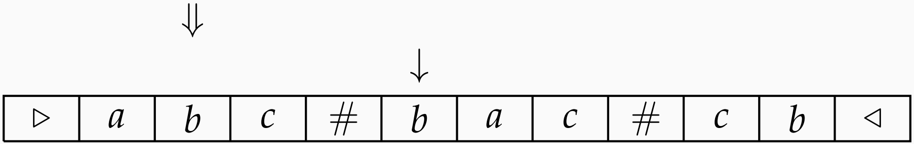

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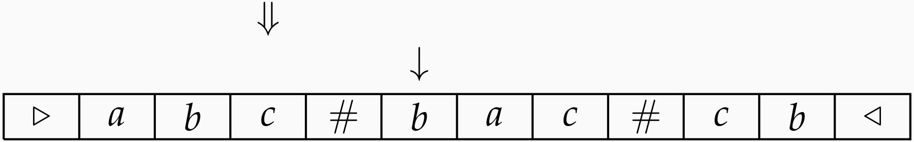

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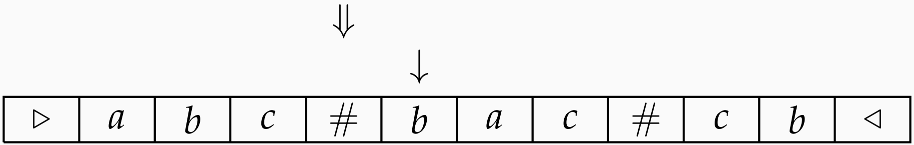

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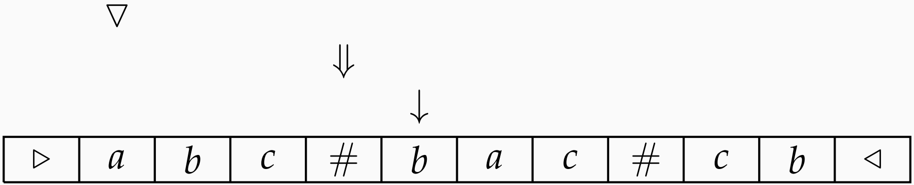

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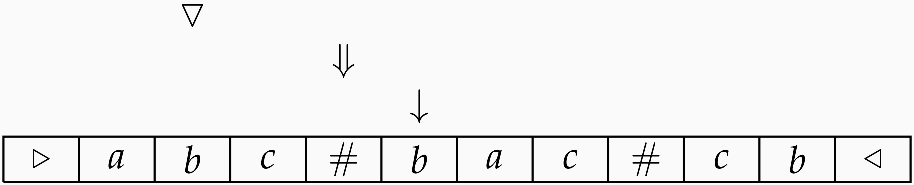
Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

"Inner squaring" $\texttt{innsq}\colon w_0\# \ldots \# w_n \longmapsto (w_0)^n\# \ldots \#(w_n)^n$



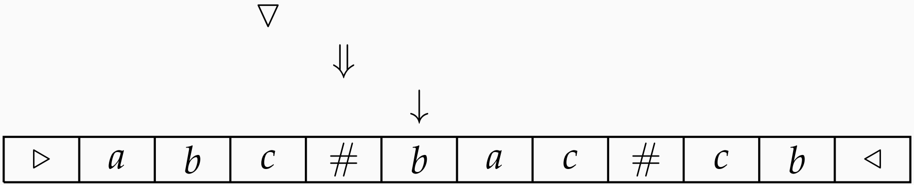
Output: *abcabc#*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

"Inner squaring" $\texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$



Output: *abcabc#*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

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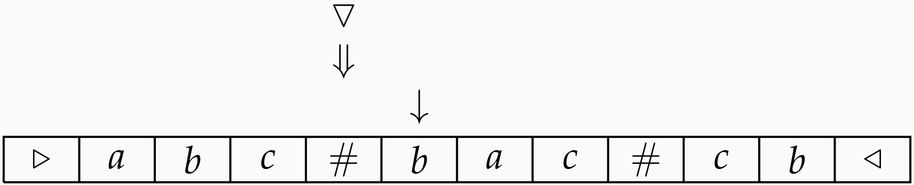

Output: $abcabc\#$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

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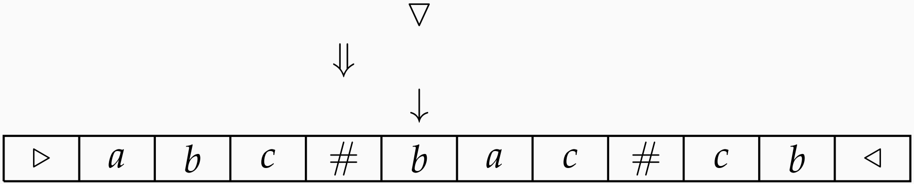


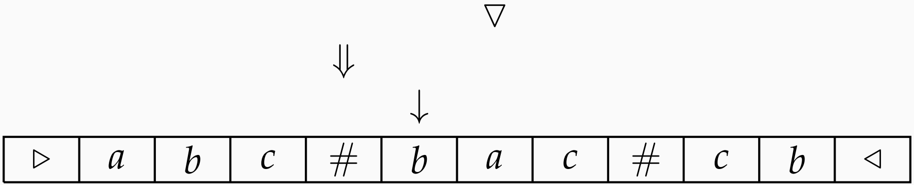
Output: $abcabc\#$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads (“pebbles”)

“Inner squaring” `innsq`: $w_0\# \dots \#w_n \longmapsto (w_0)^n\# \dots \#(w_n)^n$



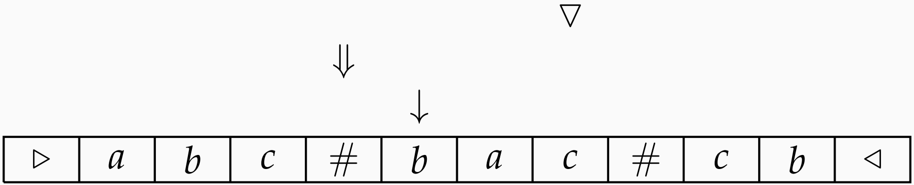
Output: $abcabc\#b$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

"Inner squaring" $\texttt{innsq}: w_0\# \ldots \#w_n \longmapsto (w_0)^n\# \ldots \#(w_n)^n$



Output: $abcabc\#ba$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

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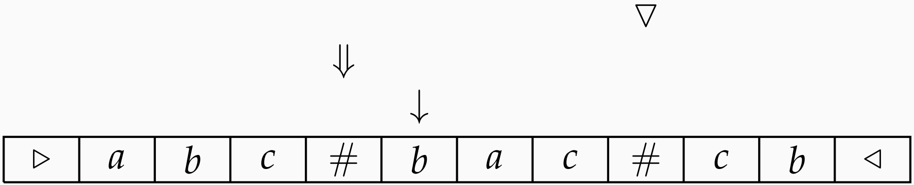
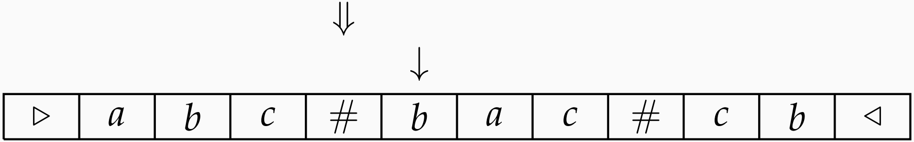


Output: $abcabc\#bac$

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

"Inner squaring" $\texttt{innsq}: w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$



Output: *abcabc#bac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

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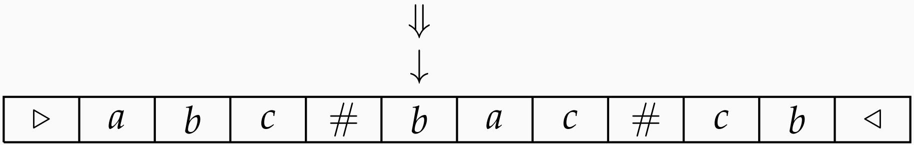

Output: *abcabc#bac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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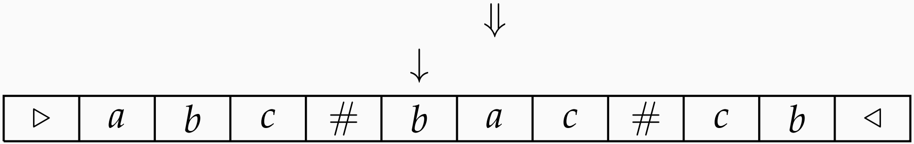

Output: *abcabc#bac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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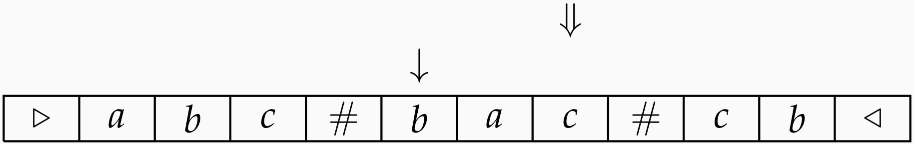

Output: *abcabc#bac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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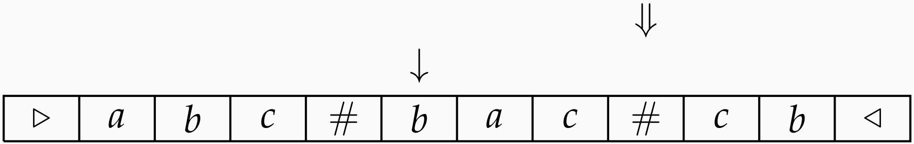

Output: $abcabc\#bac$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

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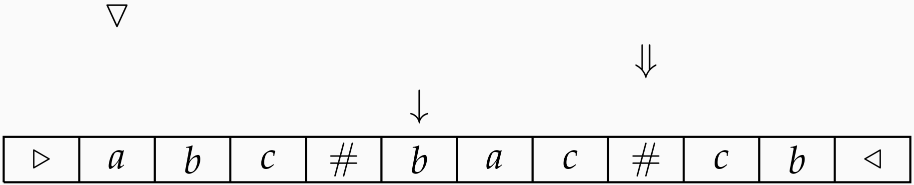

Output: *abcabc#bac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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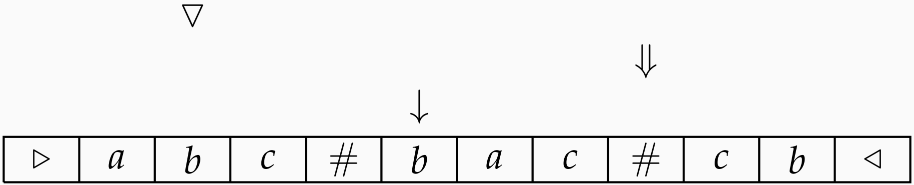

Output: $abcabc\#bac$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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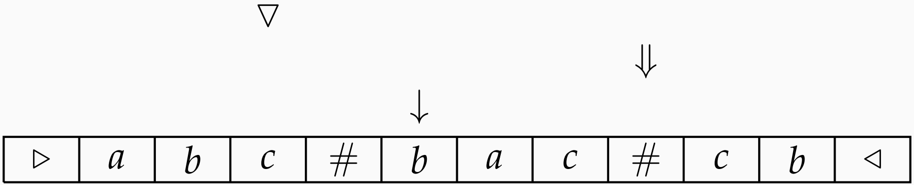

Output: $abcabc\#bac$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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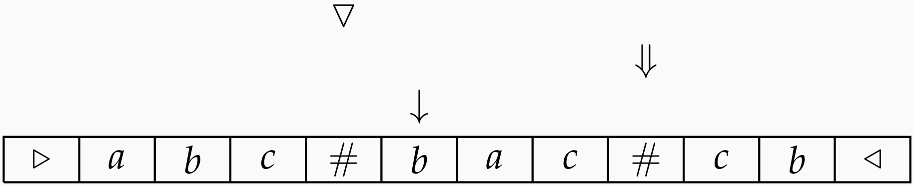


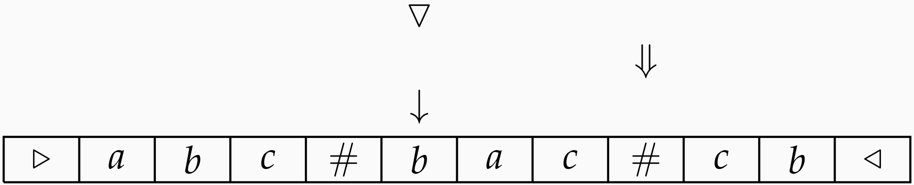
Output: *abcabc#bac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

"Inner squaring" `innsq`: $w_0\# \dots \#w_n \longmapsto (w_0)^n\# \dots \#(w_n)^n$



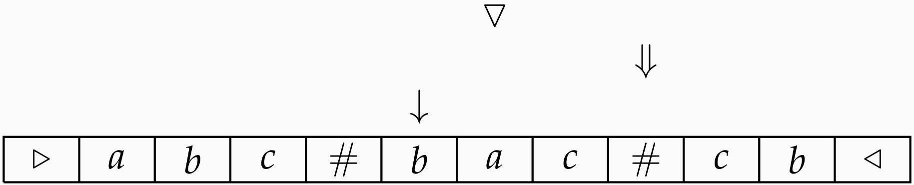
Output: $abcabc\#bac$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

"Inner squaring" $\texttt{innsq}: w_0\# \dots \#w_n \longmapsto (w_0)^n\# \dots \#(w_n)^n$



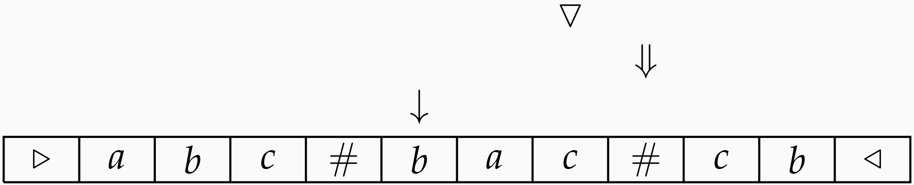
Output: *abcabc#bacb*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads (“pebbles”)

“Inner squaring” $\texttt{innsq}: w_0\# \ldots \# w_n \longmapsto (w_0)^n \# \ldots \# (w_n)^n$



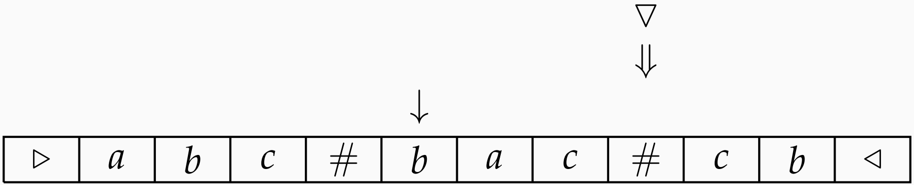
Output: *abcabc#bacba*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

"Inner squaring" `innsq`: $w_0\# \dots \#w_n \longmapsto (w_0)^n\# \dots \#(w_n)^n$



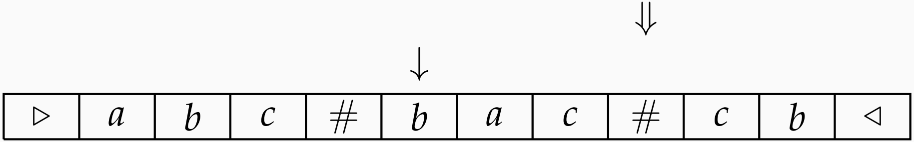
Output: *abcabc#bacbac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

"Inner squaring" `innsq`: $w_0\# \dots \#w_n \longmapsto (w_0)^n\# \dots \#(w_n)^n$



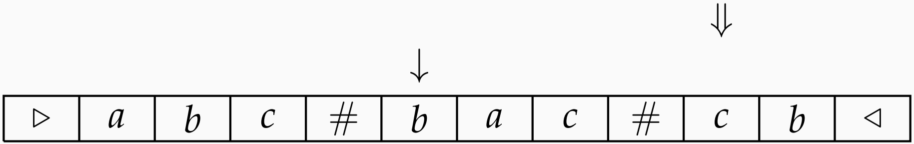
Output: *abcabc#bacbac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by *k*-pebble transducers ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

"Inner squaring" `innsq`: $w_0\# \ldots \#w_n \longmapsto (w_0)^n\# \ldots \#(w_n)^n$



Output: *abcabc#bacbac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

DFA (hidden in drawing) + *stack* of height $\leqslant k$ of heads ("pebbles")

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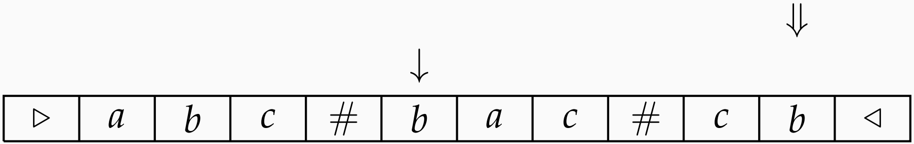

Output: *abcabc#bacbac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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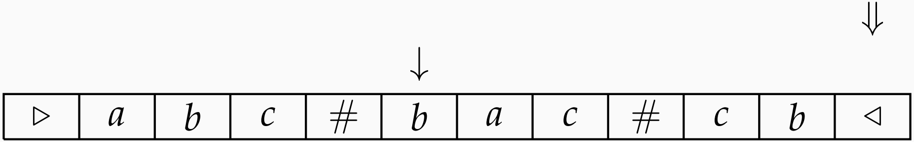

Output: *abcabc#bacbac*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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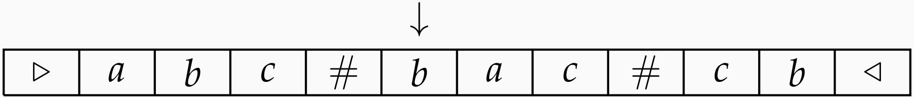

Output: *abcabc#bacbac#*

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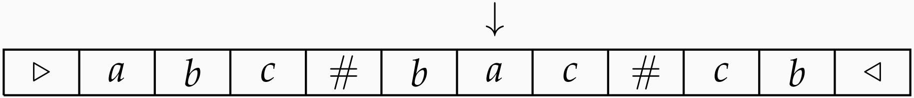

Output: $abcabc\#bacbac\#$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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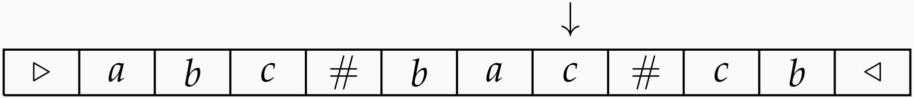

Output: $abcabc\#bacbac\#$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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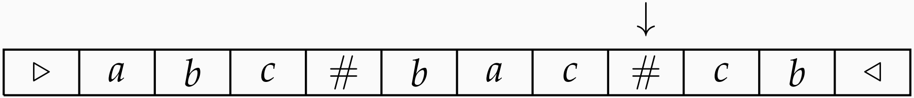

Output: *abcabc#bacbac#*

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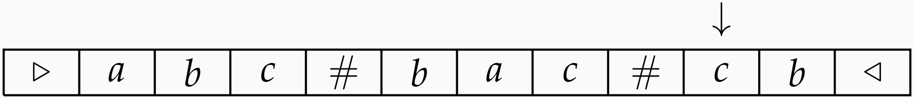

Output: *abcabc#bacbac#*

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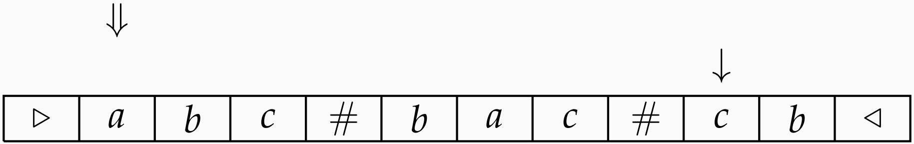

Output: *abcabc#bacbac#*

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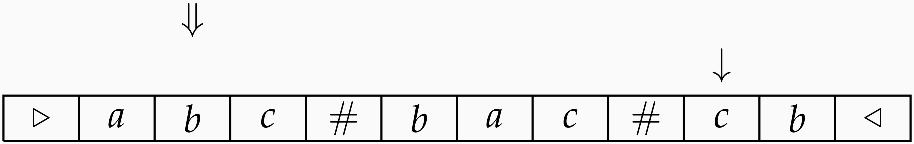

Output: *abcabc#bacbac#*

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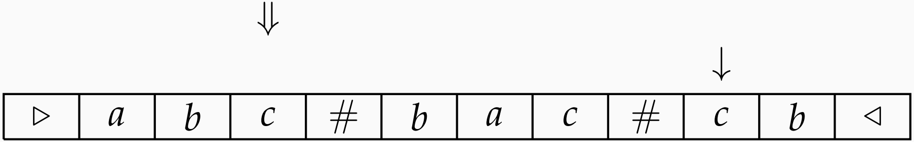

Output: $abcabc\#bacbac\#$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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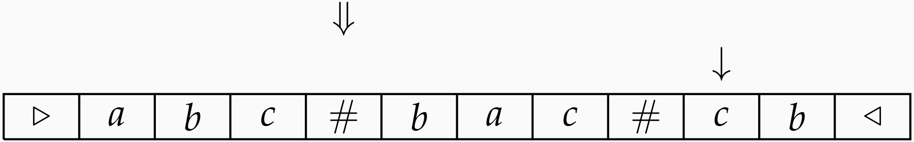

Output: *abcabc#bacbac#*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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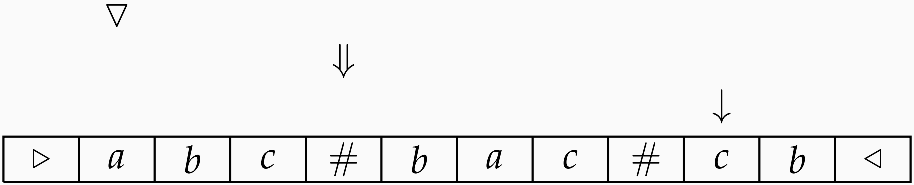

Output: $\quad abcabc\#bacbac\#$

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

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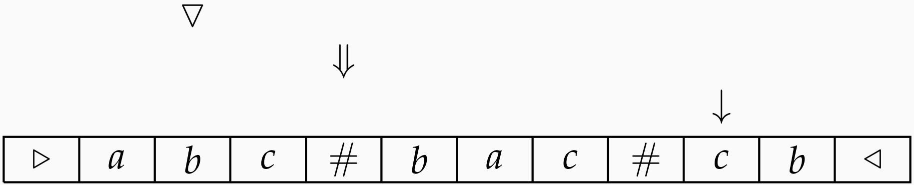

Output: *abcabc#bacbac#*

Pebble transducers [Milo, Suciu & Vianu 2000; Engelfriet & Maneth 2002]

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

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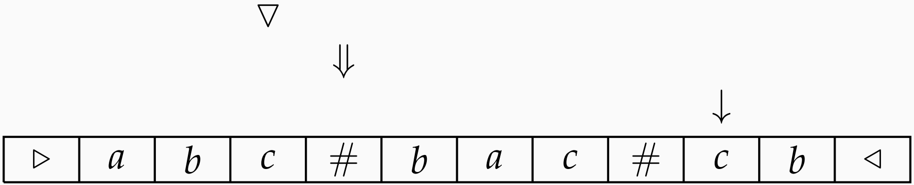

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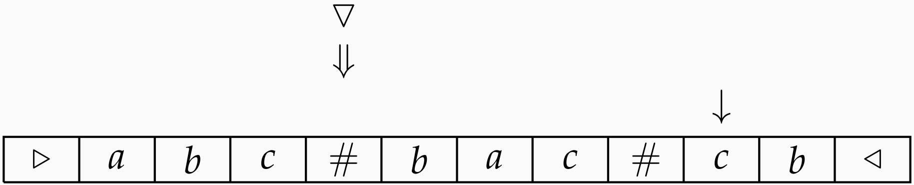

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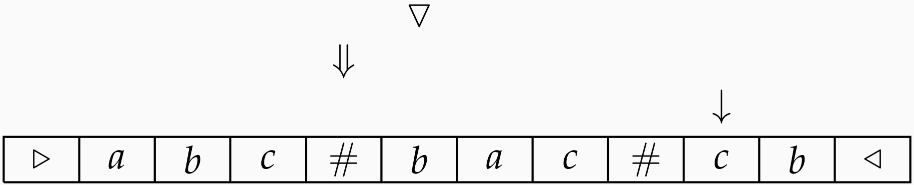

Output: $abcabc\#bacbac\#$

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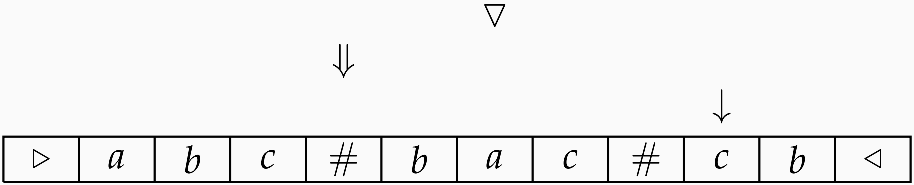

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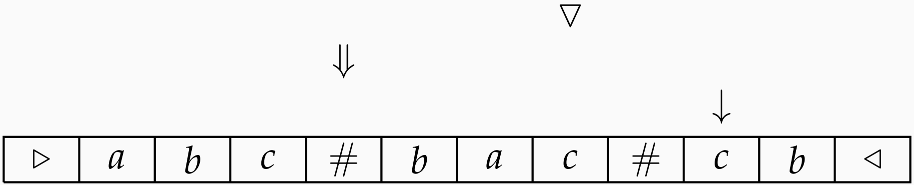

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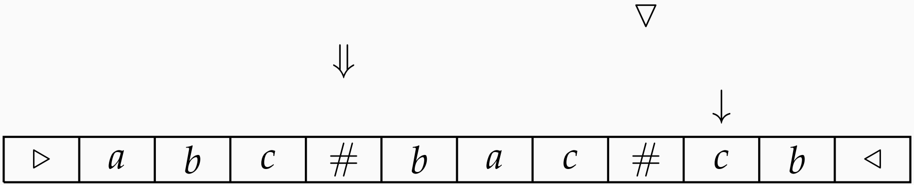

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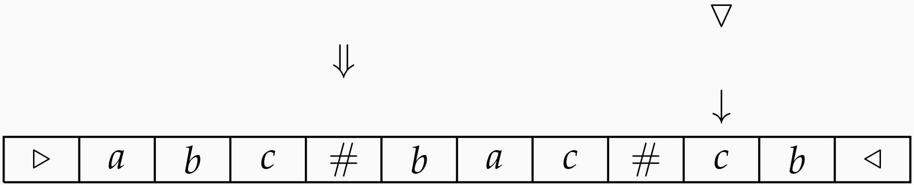

Output: *abcabc#bacbac#*

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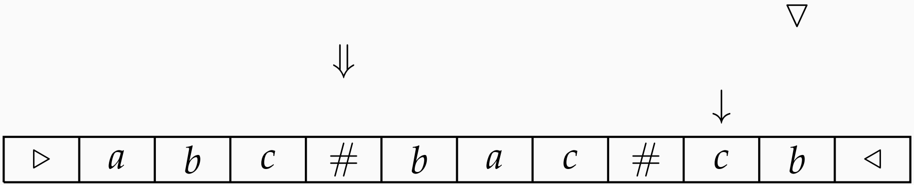

Output: *abcabc#bacbac#c*

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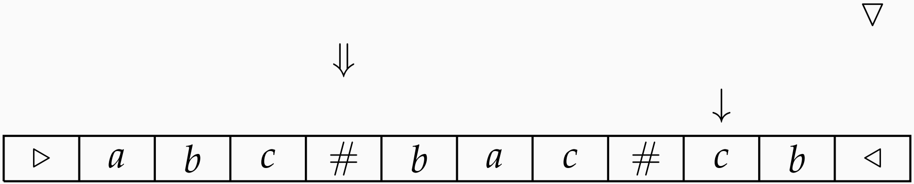

Output: $abcabc\#bacbac\#cb$

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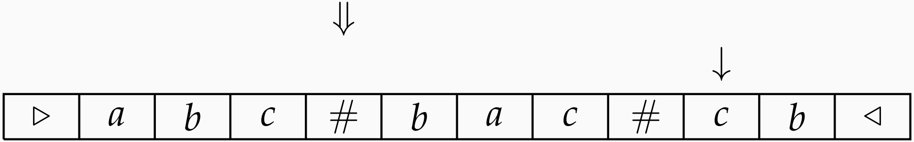

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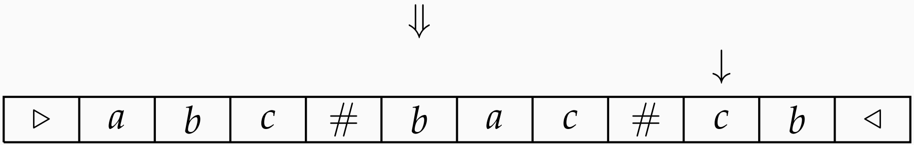

Output: *abcabc#bacbac#cb*

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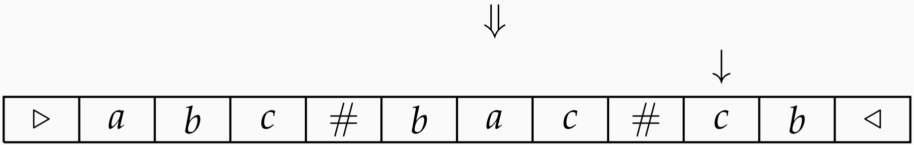

Output: $abcabc\#bacbac\#cb$

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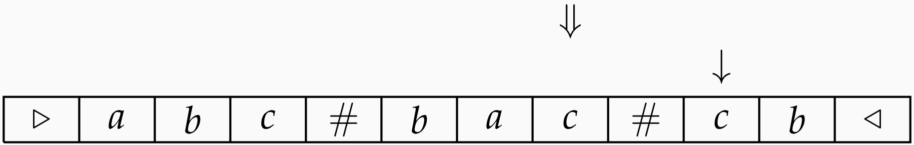

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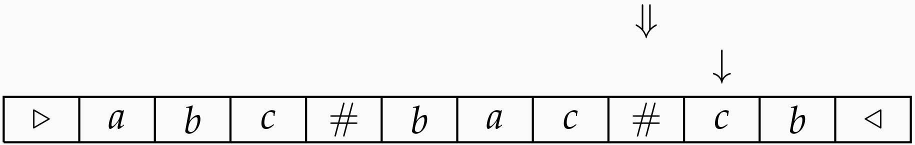

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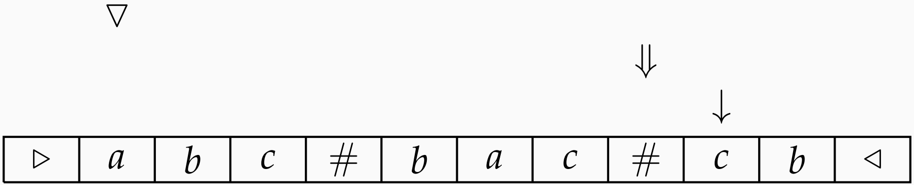

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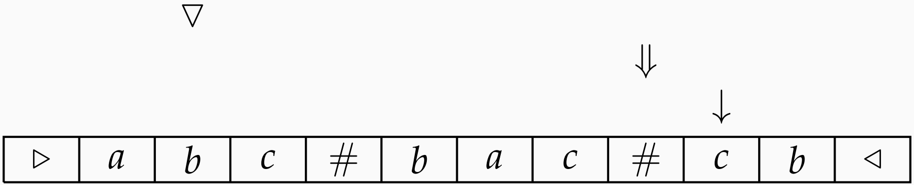

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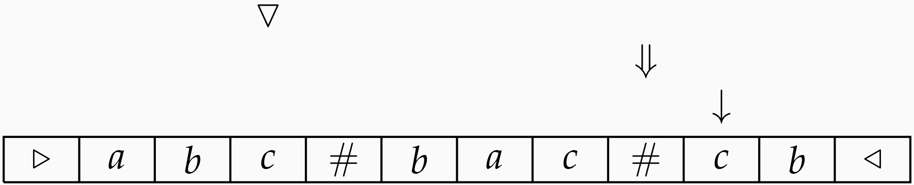

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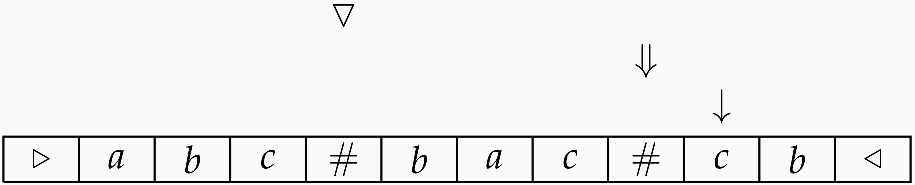

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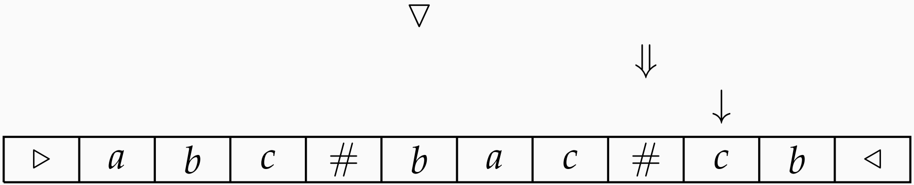

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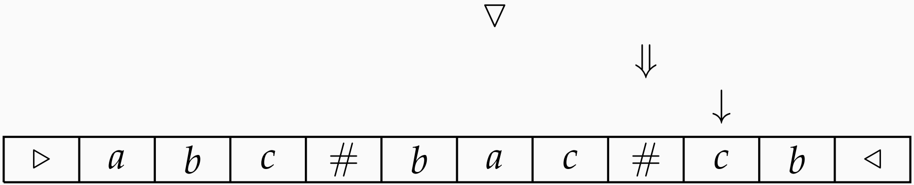

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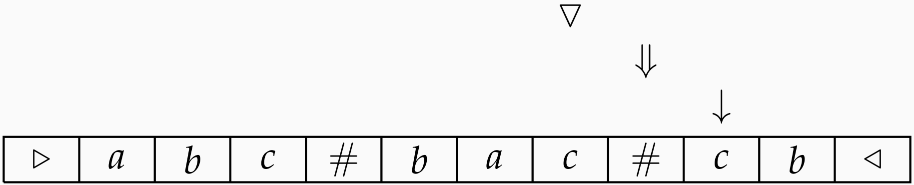

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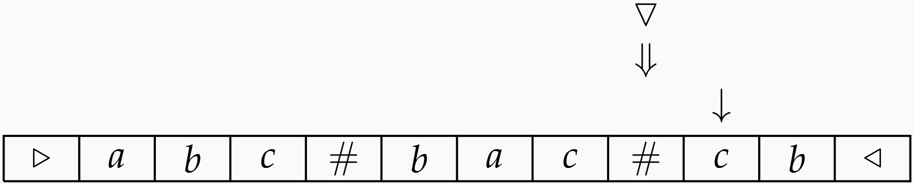

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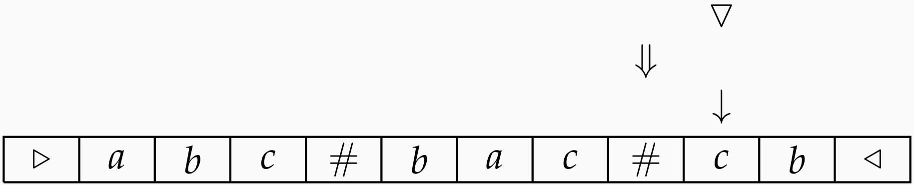

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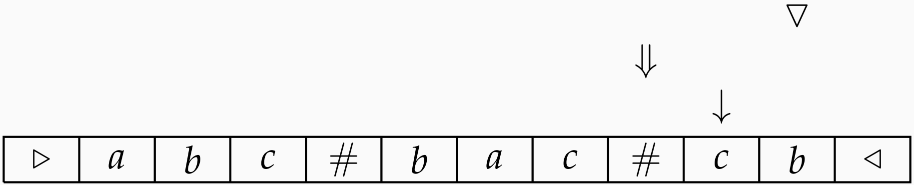

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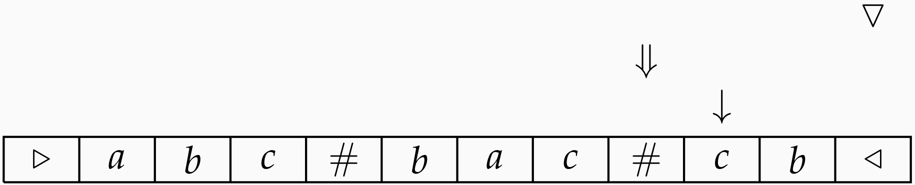

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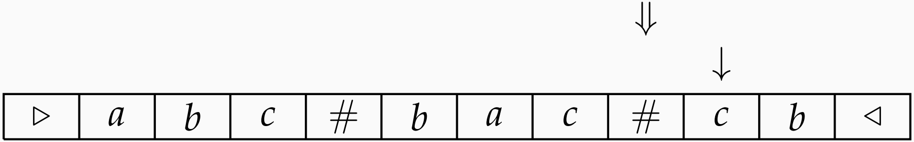

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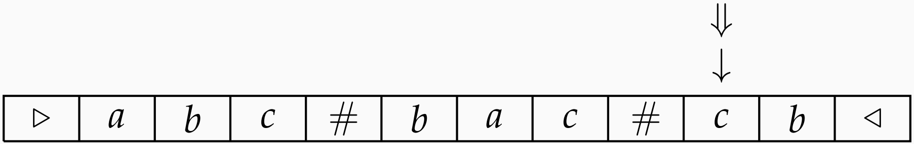

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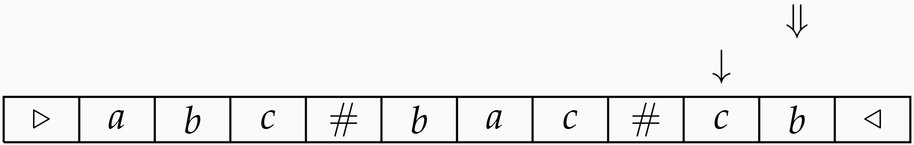

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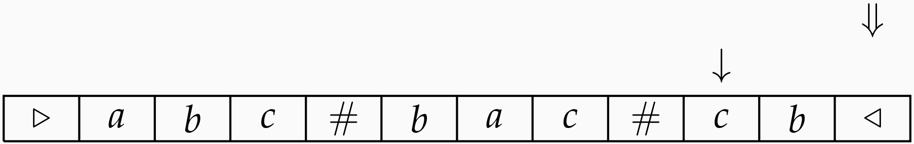

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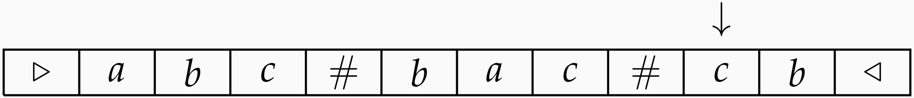

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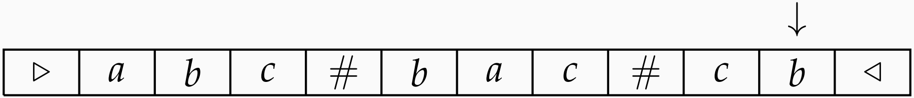

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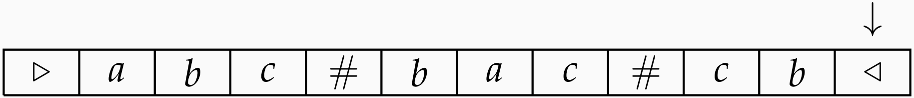

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$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $abcabc\#bacbac\#cbcb$

Not shown here: heads are *two-way* $\rightsquigarrow$ can compute e.g. `reverse`

Polyregular functions and their growth

- Closed under composition [Engelfriet & Maneth 2002; Engelfriet 2015]
  - L regular $\implies f^{-1}(L)$ regular
- Several alternative definitions in the last few years → revived interest [Bojańczyk 2018, 2023; Bojańczyk, Kiefer & Lhote 2019]
- Polynomial growth: k pebbles $\implies O(n^k)$ growth

Polyregular functions and their growth

- Closed under composition [Engelfriet & Maneth 2002; Engelfriet 2015]
  - L regular $\implies f^{-1}(L)$ regular
- Several alternative definitions in the last few years $\to$ revived interest

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What about the converse?

For $w = w_0\# \ldots \#w_n$, $|\texttt{innsq}(w)| = |(w_0)^n\# \ldots \#(w_n)^n| = O(|w|^2)$

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- Main theorem of a LICS'20 paper: $O(n^k) \implies$ computable with k pebbles
- But actually, innsq requires 3 pebbles! [Bojańczyk 2023]

Pebble non-minimization

More than that, we have:

For any $k \geq 2$ there is a polyregular f with $|f(w)| = O(|w|^2)$ that needs k pebbles.

$\longrightarrow$ the most technical proof in Bojańczyk's LICS'23 paper...

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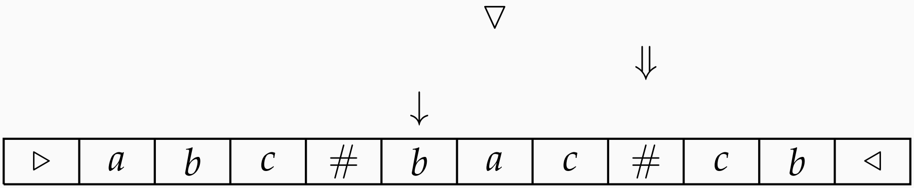
Our easier alternative, via results of Engelfriet & Maneth (2002, 2003):

Theorem (Kiefer, N. & Pradic, arXiv preprint 2023)

For any $k \geq 1$ there is a polyregular f with $|f(w)| = O(|w|^2)$ whose output language *differs from that of any* composition of k macro tree transducers.

- k-pebble $\subset$ compositions of k MTTs [EM03] $\rightarrow$ recover previous result
- this appears without the "$|f(w)| = O(|w|^2)$" in [EM02]; we adapt the proof

An obstruction to pebble minimization



Output: $abcabc\#bacba\ldots$

- when our transducer for `innsq` outputs, *the position of* ▽ *determines that of* ↓
- redundant but *necessary*: to move ⇓, you have to *pop* ▽

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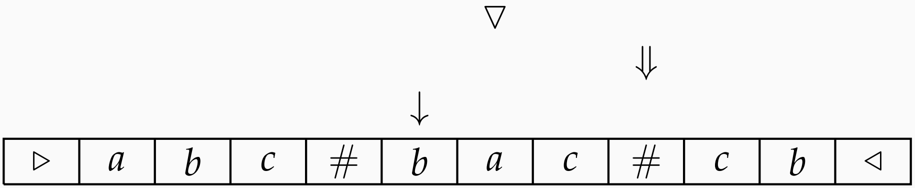

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We'd like more flexibility to only need 2 "input pointers"

⟶ another formalism: *MSO interpretations*

MSO interpretations, via an example

$\texttt{innsq}' : w_0\# \dots \# w_n \longmapsto (w_0)^n \dots (w_n)^n$ has a dim. 2 (optimal) interpretation:

$$
\begin{array}{cc}
 & acab\#abba\#c \\
\vdots & \dots \\
\# & acab \quad abba \quad c \\
\vdots & \dots \\
\# & acab \quad abba \quad c \\
\vdots & \dots
\end{array}
\longrightarrow (acab)(acab)(abba)(abba)(c)(c)
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$$

- $\varphi_a(x_1, x_2) = a(x_1) \wedge \#(x_2)$ (in general: Monadic Second-Order logic formula)
- $\varphi_\prec(x_1, x_2, y_1, y_2) = \exists x_3, y_3.$ which begin blocks containing resp. x_1, y_1 and $(x_3, x_2, x_1) < (y_3, y_2, y_1)$ lex. $\longrightarrow$ pebbles $\downarrow, \Downarrow, \triangledown$

Polyregular function of growth $O(n^k)$ = MSO interpretation of dimension k

This is indeed another definition of polyregular functions:

Theorem (Bojańczyk, Kiefer & Lhote 2019)

String-to-string MSO interpretations $\iff$ *pebble transducers*

- This is black magic

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MSO interpretations of dimension k $\iff$ *MSO interpretations* of growth $O(n^k)$

- Reduces easily to a lemma on the growth of MSO *queries* $\varphi(x_1, \dots, x_k)$
  - Bojańczyk LICS'23: proof using factorization forests
  - Kiefer, N. & Pradic (originally Lhote's idea):
    derive the lemma from well-known properties of $\mathbb{N}$-weighted automata

Conclusion

Polyregular functions = "feasible finite-state string functions"

Several equivalent definitions: pebble transducers,
Monadic Second-Order interpretations, ...

- f polyregular $\wedge |f(w)| = O(|w|^k) \iff f$ defined by MSOI of dimension k
- for all k, polyregular $\cap$ growth $O(n^2) \not\Rightarrow$ computable with k pebbles

(but trivially, k pebbles $\implies$ growth $O(n^k)$)

Example: "inner squaring"

$w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$ has growth $O(n^2)$ but requires 3 pebbles

Shown by Bojańczyk; we reprove all this by quick applications of old technology

("we" = Sandra Kiefer, N. & Cécilia Pradic; "old technology": moral of the story: go read Engelfriet!)

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Thanks for your attention!

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