BV and Pomset Logic are not the same

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What is this about?

Pomset Logic (PL) and *system BV*: 2 logics over the same formulas

A two-decades-old conjecture

These logics are equivalent, i.e. prove the same formulas.

It was known that $(\mathrm{BV} \vdash A) \implies (\mathrm{PL} \vdash A)$.

Our result: refuting the conjecture

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The classical sequent calculus LK

An usual proof system for classical logic:

- Identity and cut rules
- Logical rules:

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Remove contraction and weakening $\rightarrow$ *Multiplicative Linear Logic* (MLL)

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$$A, B ::= a \mid a^\perp \mid A \otimes B \mid A \parr B$$

Involutive negation *defined* by De Morgan rules:

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$\longrightarrow$ semantics may suggest extensions to the logic

Extensions to Multiplicative Linear Logic

The denotational semantics of MLL in (hyper)coherence spaces suggest:

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Two conservative extensions of MLL+Mix with $A, B ::= \cdots \mid A \lhd B$

- Pomset Logic (Christian Retoré, early 1990s) – based on *proof nets*
- System BV (Alessio Guglielmi, late 1990s) – the first system based on *deep inference*

Motivations

Guglielmi 2007, *A System of Interaction and Structure* (emphasis mine):

> **It is still open whether the logic in this paper, called *BV*, is the same as pomset logic. We conjecture that it is actually the same logic**, *but one crucial step is still missing, at the time of this writing, in the equivalence proof. This paper is the first in a planned series of 3 papers dedicated to BV.* [...] *In the 3rd part, some of my colleagues will hopefully show the equivalence of BV and pomset logic,* [...]

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There are formal arguments showing that "traditional sequent calculi cannot express BV"
[Tiu 2006]

A glance at deep inference

A methodology originally introduced for BV; many other successes in past 2 decades (e.g. cut-free proofs for modal logics)

Deep inference = unary rules applied to subformulas of arbitrary depth:

$$\text{inference rule } \frac{A}{B} \qquad \rightsquigarrow \qquad \text{instances } \frac{S[A]}{S[B]} \text{ for any context } S$$

e.g.

$$\frac{A \parr (B \otimes (C \parr D))}{A \parr ((B \otimes C) \parr D)}$$

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(compare with rewriting systems, or functoriality in categorical logic)

Deep inference for MLL+Mix and BV

- Identity rule: $\overline{a, a^{\perp}}$
- MLL+Mix: rules for assoc/comm. of $\otimes, \parr$ + unitality ($A \otimes \mathbf{I} \equiv A \parr \mathbf{I} \equiv A$) +

$$\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \text{(where } A, B, C \text{ may be equal to } \mathbf{I}\text{)}$$

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- BV: MLL+Mix rules for associativity/unitality of $\lhd$ (*not* commutativity!) +

$$\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \qquad \text{(where } A, B, C, D \text{ may be equal to } \mathbf{I}\text{)}$$

Proof nets for Multiplicative Linear Logic

The proof system for Pomset Logic extends the graphical syntax of MLL *proof nets*

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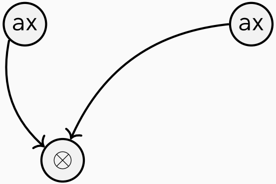

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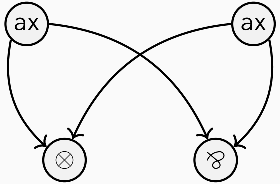

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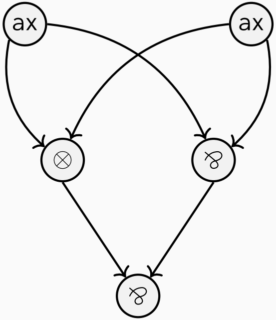

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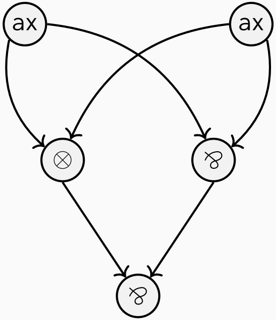


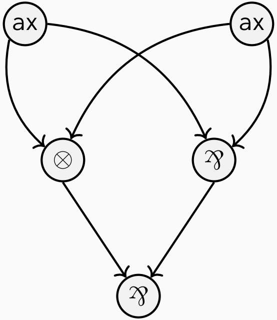
No distinction between $\otimes$ and $\parr$ $\longrightarrow$ not all graphs correspond to correct proofs

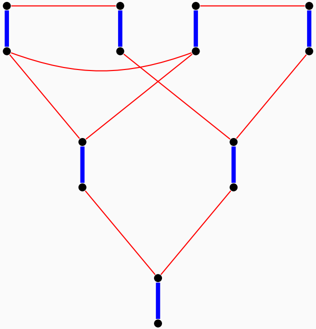
$\longrightarrow$ need a *correctness criterion*

In addition to Pomset Logic, Retoré also invented in the 1990s...

A translation MLL+Mix proof nets → graphs equipped with *perfect matchings*

(for linear logicians: reformulation of Danos–Regnier switching criterion)





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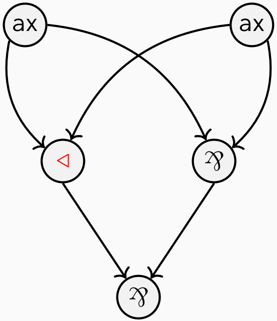

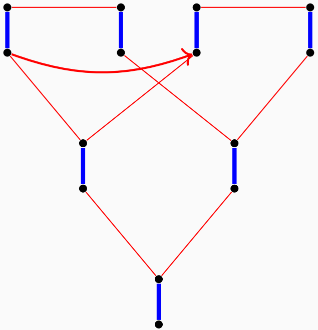

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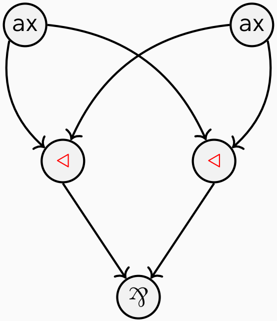

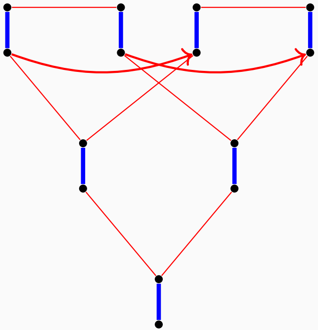

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Complexity of proof net correctness

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This led us to a surprising realization:

Theorem (not in the paper, to appear in the upcoming journal version)

Provability in pomset logic is strictly harder *than in BV unless* $\mathsf{NP} = \mathsf{coNP}$.

(*more precisely:* Σ_2^{p}*-complete vs* NP*-complete*)

- In BV, the length of proofs is polynomially bounded
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$\longrightarrow$ Suddenly, Guglielmi's conjecture looked less plausible...

Conclusion

Retoré's *Pomset Logic* (PL) and Guglielmi's *system BV*: 2 logics over the same formulas, from the 1990s, conservatively extending Multiplicative Linear Logic with Mix

Guglielmi's two-decades-old conjecture

$(\mathrm{BV} \vdash A) \iff (\mathrm{PL} \vdash A)$ (the left-to-right implication was known)

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Thanks for your attention!

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