Comparison-free polyregular functions a.k.a. “polyblind functions”

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joint work with Pierre Pradic (Lecturer, University of Swansea)

December 14th, 2021 — Trends in Transformations, FSTTCS satellite workshop (Online)

Reminder: deterministic two-way transducers (2DFT)

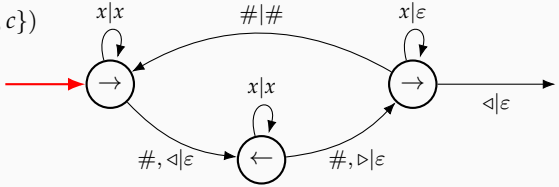
Two-way transducers: executive summary **(recall Sarah Winter's talk)**

Finite set of states + bidirectional reading head + output produced from left to right

Example:

$$\begin{array}{lccl} \texttt{mapPalin}: & \{a,b,c,\#\}^* & \longrightarrow & \{a,b,c,\#\}^* \\ & w_1\#\ldots\#w_n & \longmapsto & w_1 \cdot \texttt{reverse}(w_1)\#\ldots\#w_n \cdot \texttt{reverse}(w_n) \end{array}$$

$(x \in \{a,b,c\})$



Output:

▷	a	b	c	#	b	a	c	#	c	a	◁

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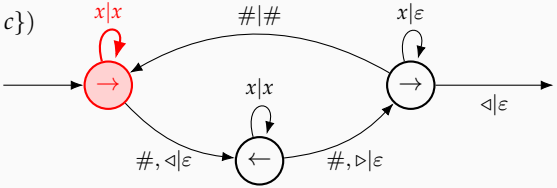

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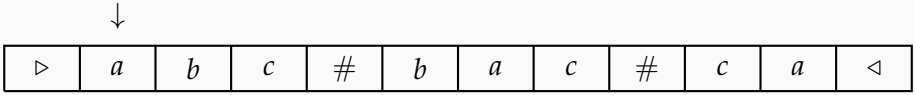

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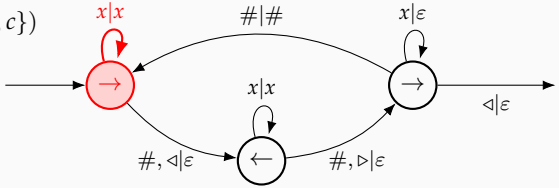
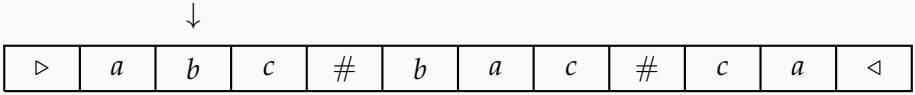


Output:

a



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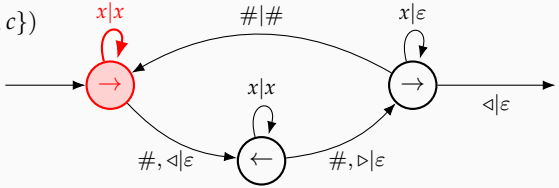
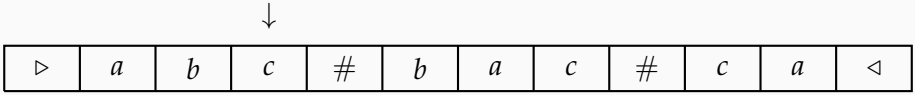


Output:
ab



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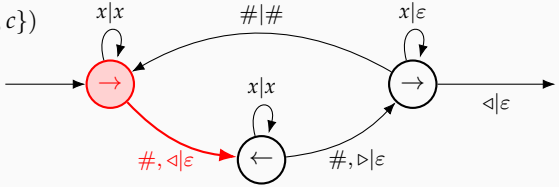
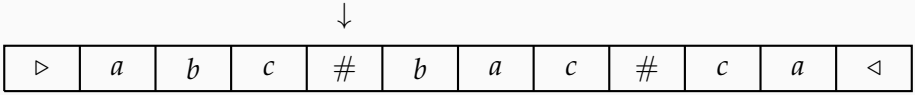


Output:
abc



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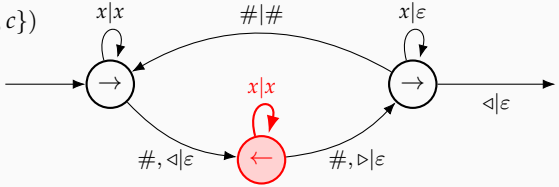
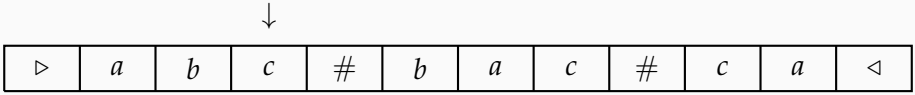


Output:
abc



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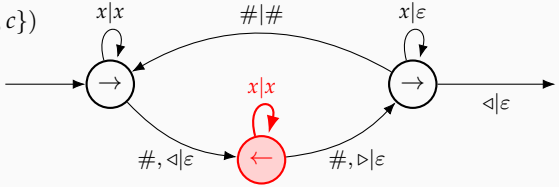

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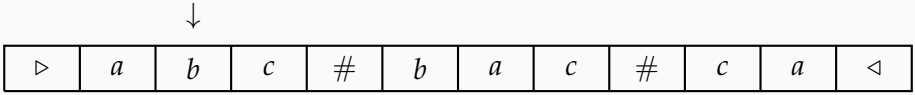
abcc



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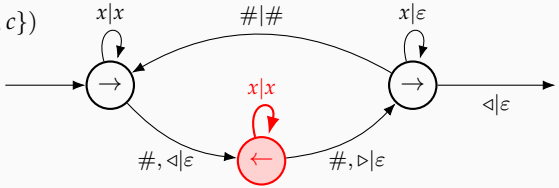
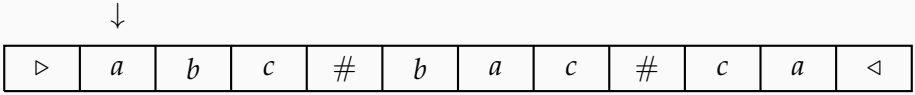


Output:
abccb



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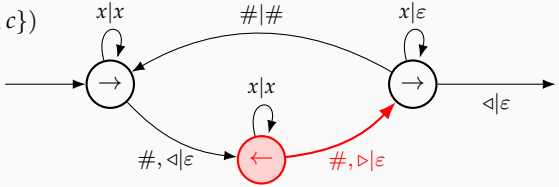
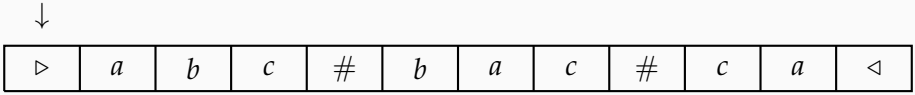


Output:
abccba



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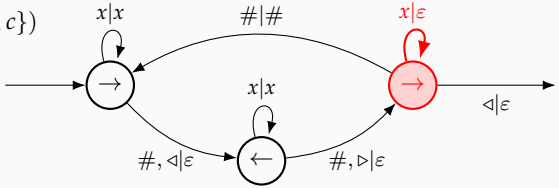

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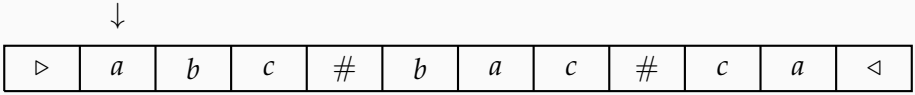

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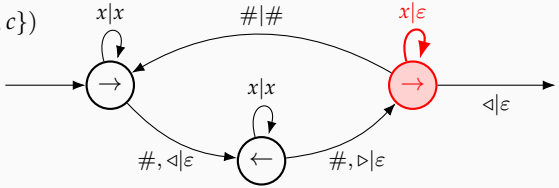

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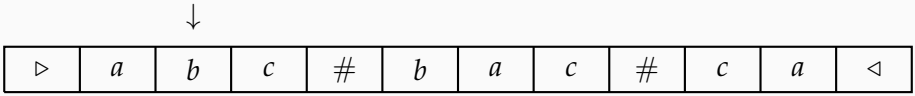

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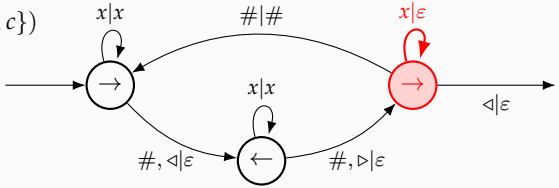

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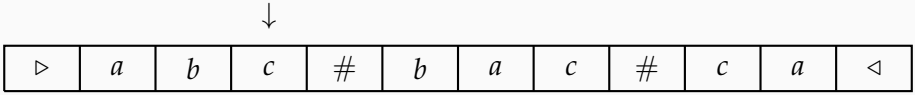

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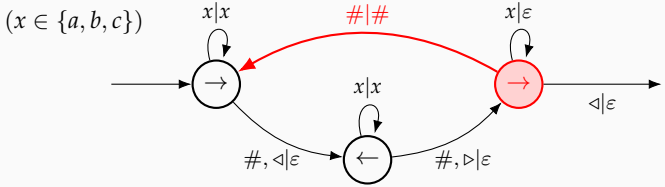

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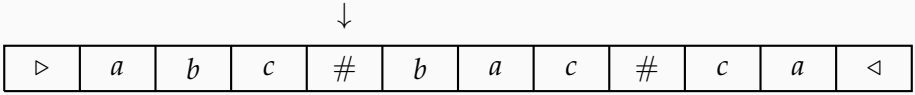

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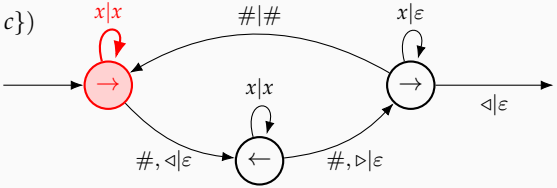

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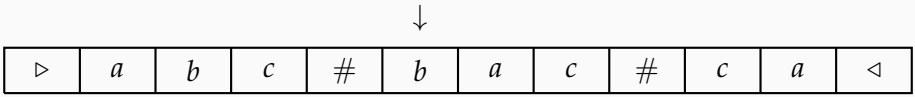
abccba#



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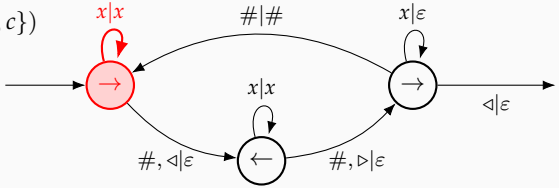

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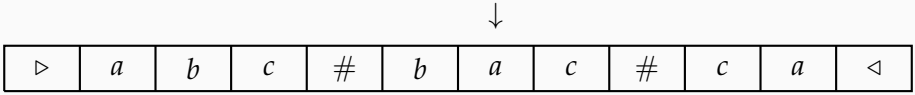
abccba#b



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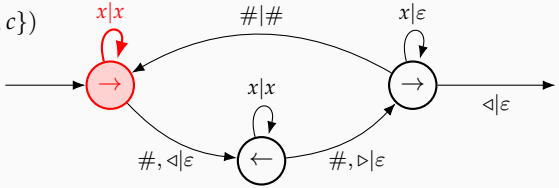

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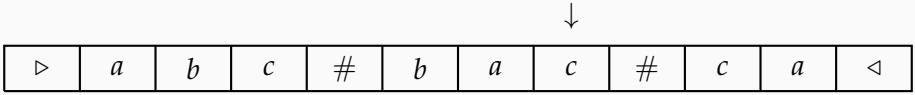

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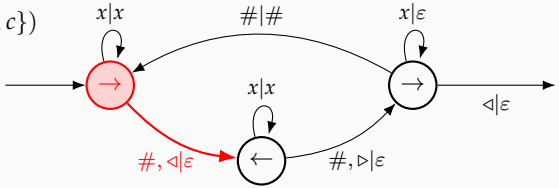

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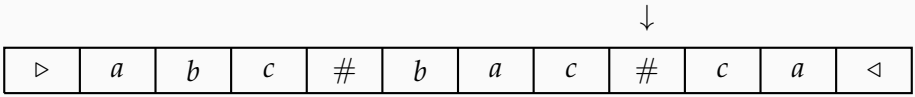

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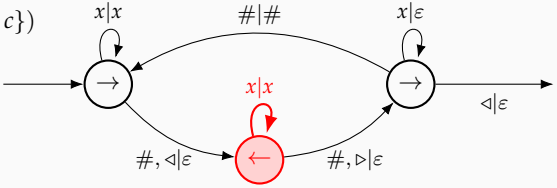

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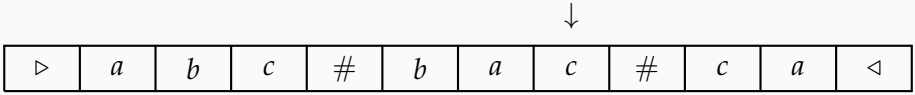

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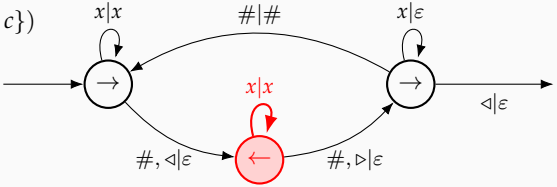

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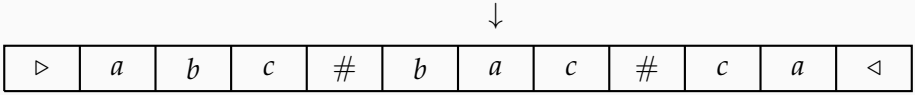

Reminder: deterministic two-way transducers (2DFT)

Two-way transducers: executive summary **(recall Sarah Winter's talk)**

Finite set of states + bidirectional reading head + output produced from left to right

Example:

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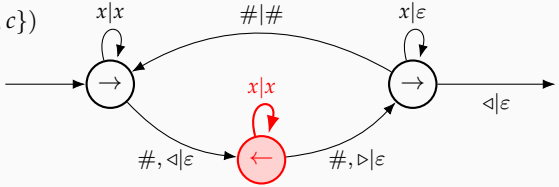

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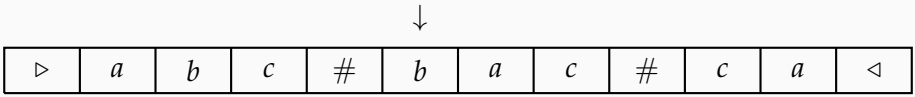

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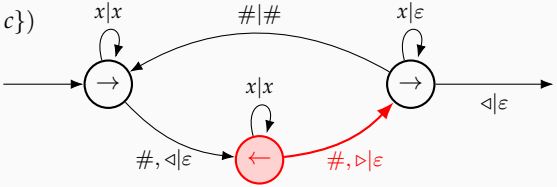

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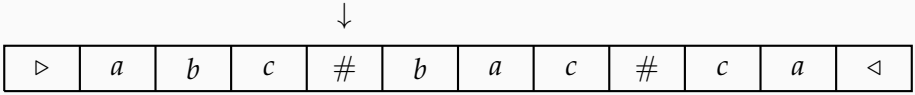

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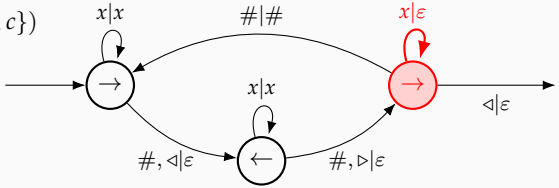

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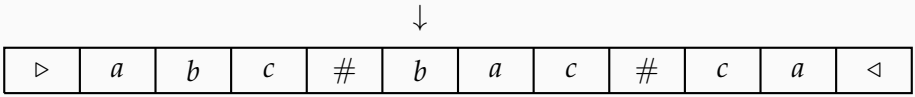

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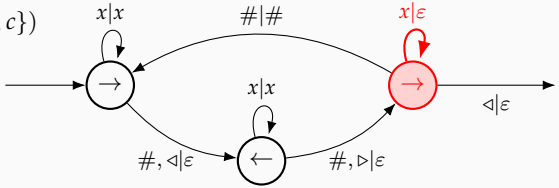

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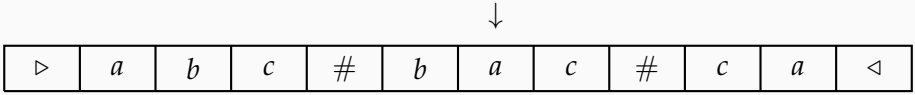

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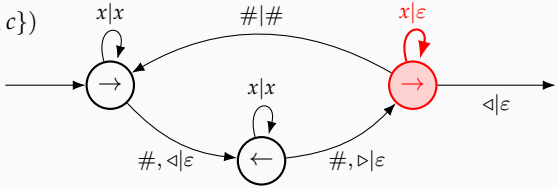

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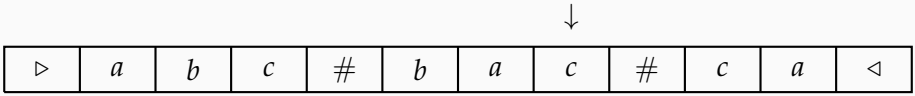

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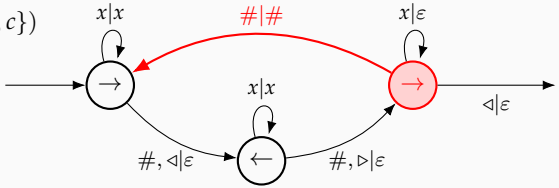

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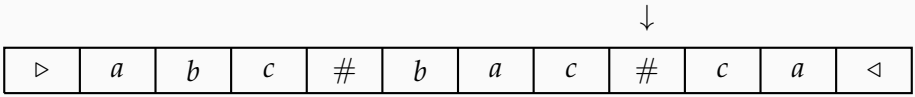

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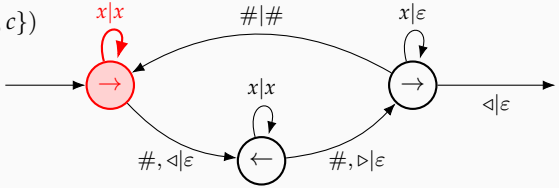

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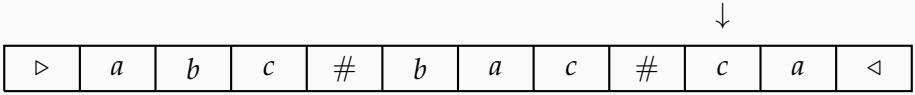

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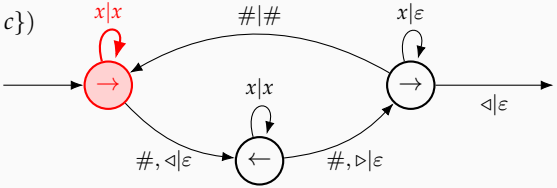

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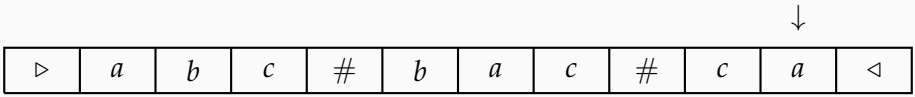

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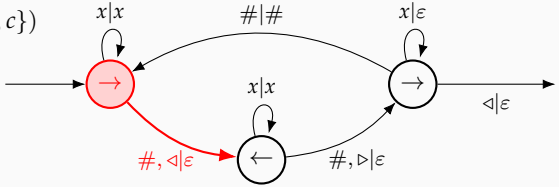

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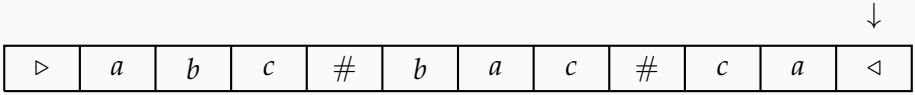

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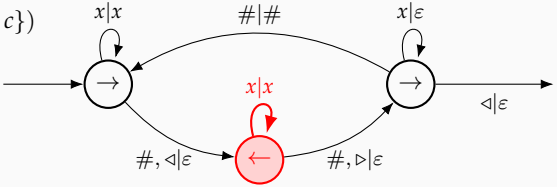

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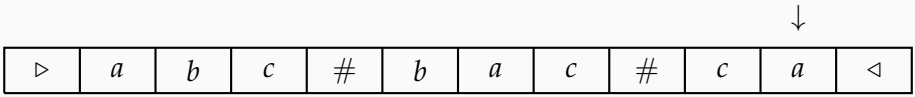

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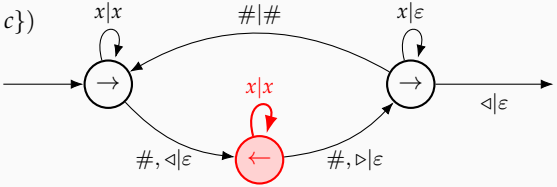

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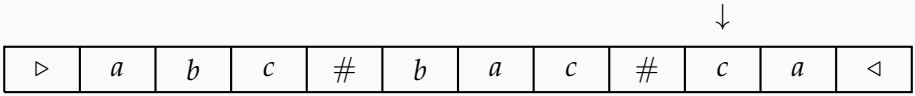

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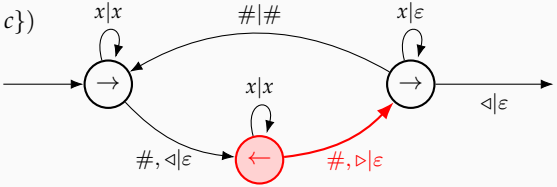

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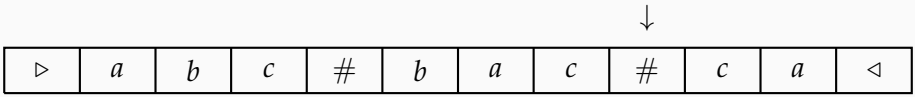

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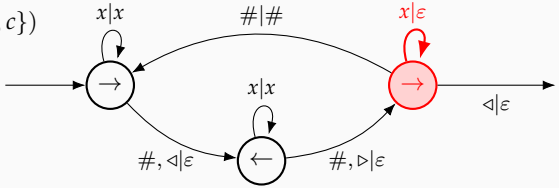

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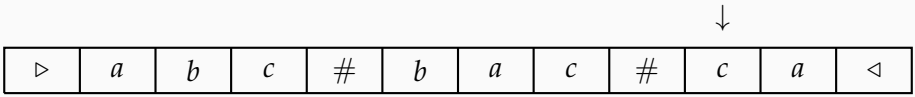

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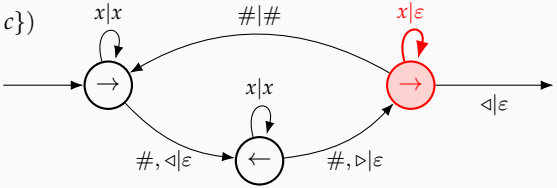

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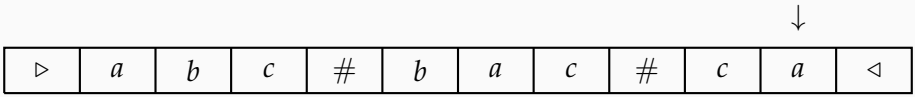

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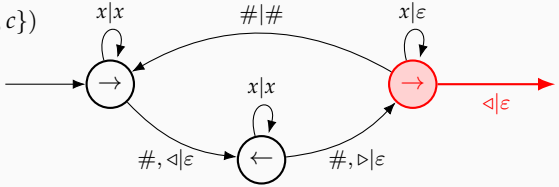

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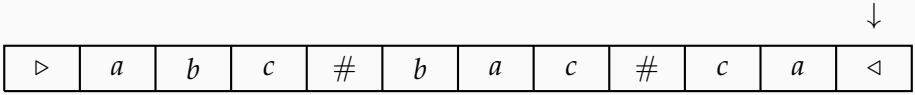

Regular functions

Functions $\Sigma^* \to \Gamma^*$ definable by 2DFTs are called **regular (string) functions**

($\supsetneq$ rational functions, cf. Sarah's talk)

Properties of regular functions

- Linear growth: $|f(w)| = O(|w|)$
- Closed under composition (if $f : \Gamma^* \to \Sigma^*$ and $g : \Sigma^* \to \Pi^*$ are regular then so is $g \circ f$)
- L regular $\implies f^{-1}(L)$ regular

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Alternative characterizations

- Via Monadic Second-Order logic (MSO transductions *à la* Courcelle)
- Copyless streaming string transducers (one-way machine model)
- Various functional programming or regexp-like (declarative) formalisms,
  e.g. "regular list functions" [Bojańczyk, Daviaud & Krishna 2018]
- (recent work of ours) Minimal linear λ-calculus and Church encodings
  $\longrightarrow$ see my PhD thesis *Implicit automata in linear logic and categorical transducer theory*
  `https://nguyentito.eu/thesis.pdf`

Polyregular functions

- A larger class of string-to-string transductions (regular $\subsetneq$ polyregular)
- Garnered significant attention recently, starting with [Bojańczyk 2018]

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Characterizations [Bojańczyk 2018; Bojańczyk, Kiefer & Lhote 2019]

- Multidimensional MSO interpretations
- Imperative nested loop programs
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  simply typed λ-calculus + primitive list type w/ some list manipulation primitives
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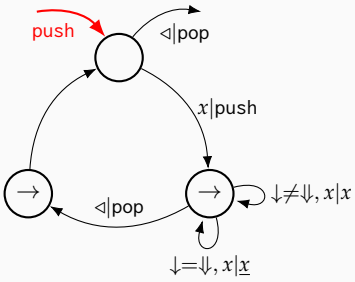
- **k-pebble string-to-string transducers** [Milo, Suciu & Vianu 2000]

k-pebble transducers: executive summary

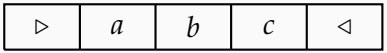
Finite set of states + a stack of two-way reading heads of height $\leq k$

- Heads can be moved, pushed, popped
- Arbitrary comparisons between heads in the stack
- 1-pebble transducers $\cong$ 2DFTs

Example: "squaring with underlining" ($k = 2$)



$$\texttt{squaring}: \quad \Sigma^* \to (\Sigma \cup \underline{\Sigma})^* \qquad aab \mapsto \underline{a}aba\underline{a}baa\underline{b}$$



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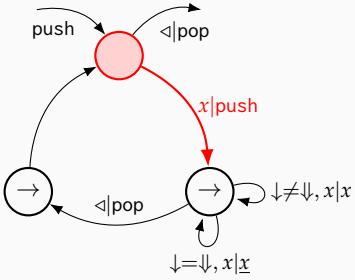

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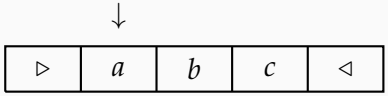

output =

k-pebble transducers: executive summary

Finite set of states + a stack of two-way reading heads of height $\leq k$

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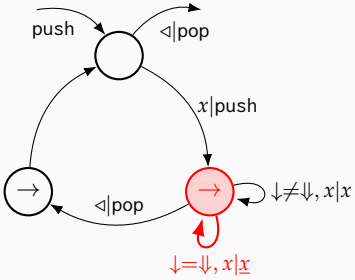

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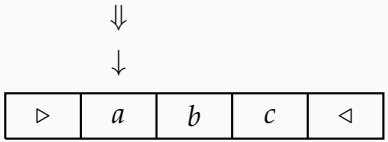

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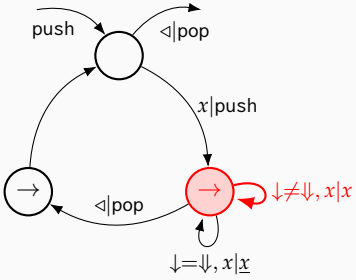

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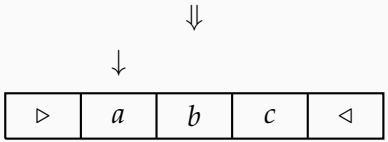

output $= \underline{a}$

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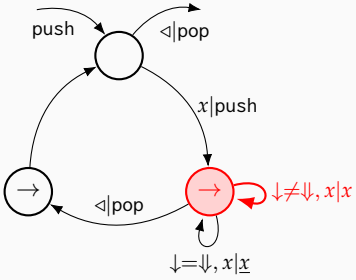

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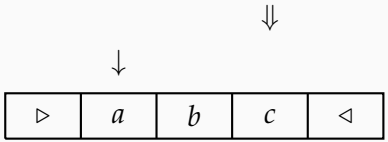

output $= \underline{a}b$

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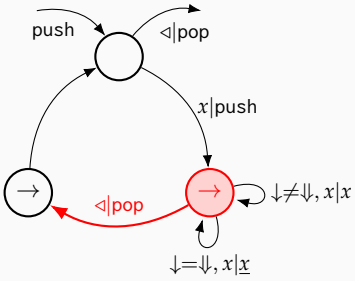

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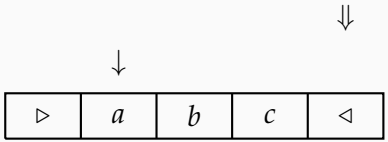

output $= \underline{a}bc$

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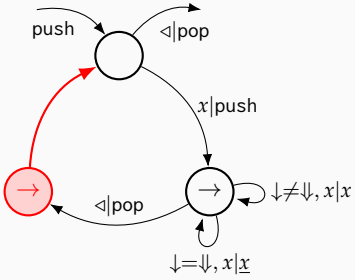

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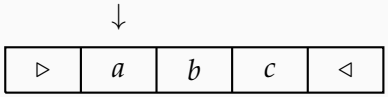

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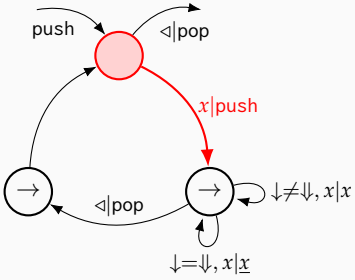

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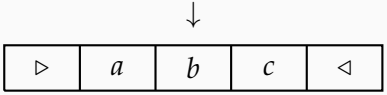

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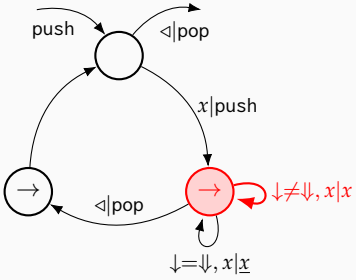

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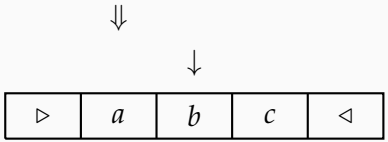

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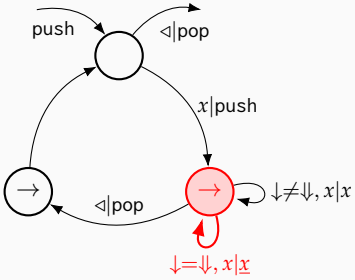

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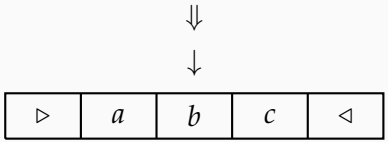

output $= \underline{a}bca$

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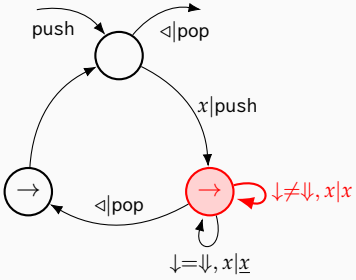

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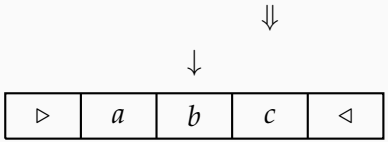

output $= \underline{a}bca\underline{b}$

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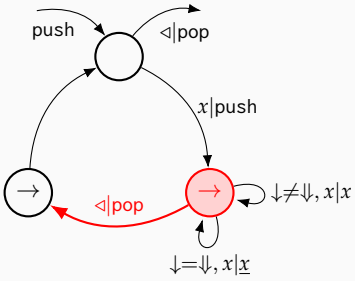

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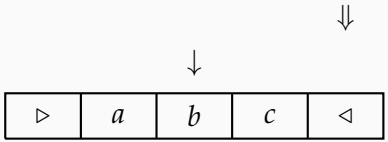

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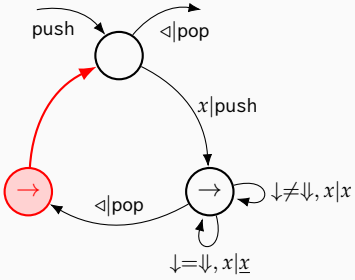

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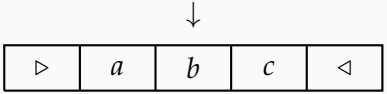

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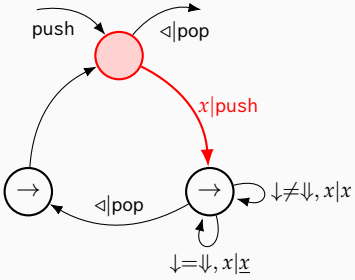

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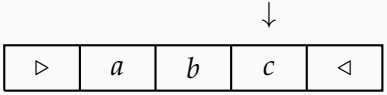

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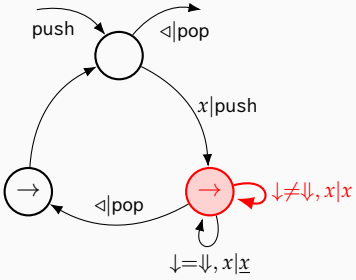

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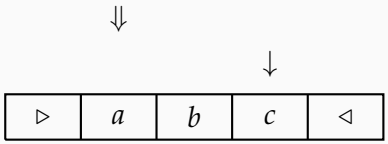

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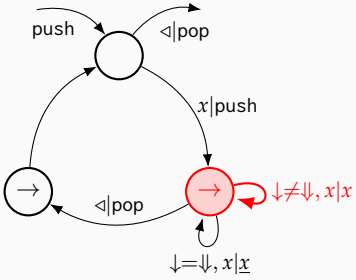

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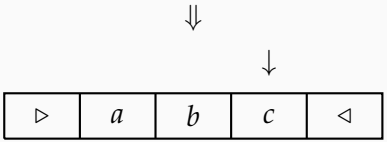

output $= \underline{a}bca\underline{b}ca$

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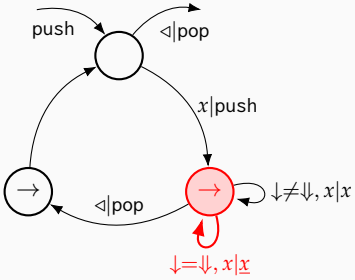

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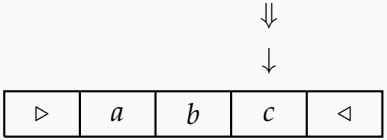

output $= \underline{a}bca\underline{b}cab$

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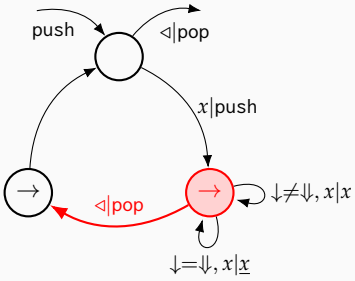

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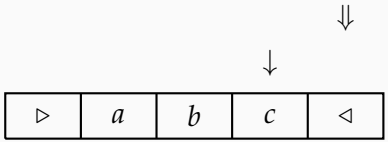

output $= \underline{a}bca\underline{b}cab\underline{c}$

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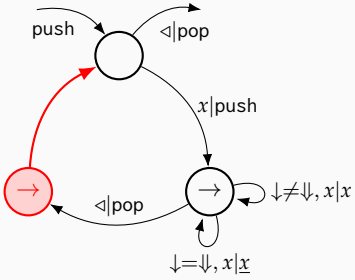

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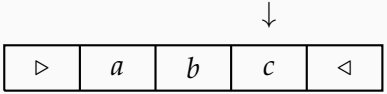

output $= \underline{a}bca\underline{b}cab\underline{c}$

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Finite set of states + a stack of two-way reading heads of height $\leq k$

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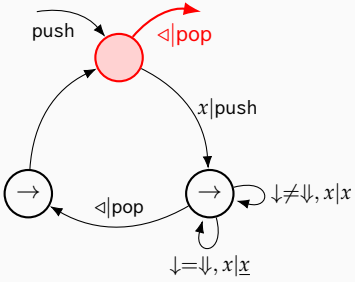

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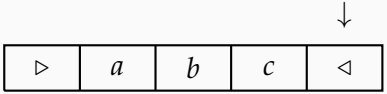


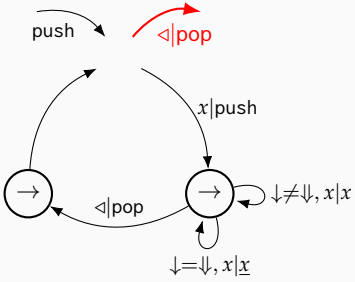
output $= \underline{a}bca\underline{b}cabc\underline{c}$

k-pebble transducers: executive summary

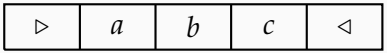
Finite set of states + a stack of two-way reading heads of height $\leq k$

- Heads can be moved, pushed, popped
- Arbitrary comparisons between heads in the stack $\rightarrow$ **what if we remove them?**
- 1-pebble transducers $\cong$ 2DFTs

Example: "squaring with underlining" ($k = 2$)



$$\texttt{squaring}: \quad \begin{array}{ccc} \Sigma^* & \rightarrow & (\Sigma \cup \underline{\Sigma})^* \\ aab & \mapsto & \underline{a}aba\underline{a}baa\underline{b} \end{array}$$



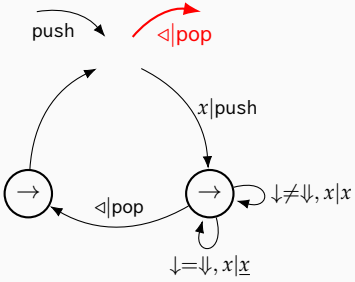
output $= \underline{a}bca\underline{b}cabc\underline{c}$

k-pebble transducers: executive summary

Finite set of states + a stack of two-way reading heads of height $\leq k$

- Heads can be moved, pushed, popped
- Arbitrary comparisons between heads in the stack $\rightarrow$ **what if we remove them?**
- 1-pebble transducers $\cong$ 2DFTs

Example: "squaring with underlining" ($k = 2$) **seems to require comparisons**



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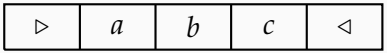


output $= \underline{a}bca\underline{b}cabc\underline{c}$

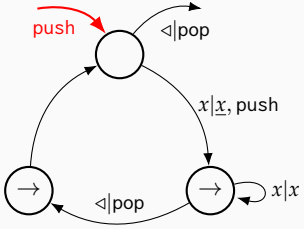
Blind transducers (a.k.a. "comparison-free pebble transducers")

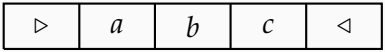
Polyblind functions = computed by blind transducers

Pebble transducers where the comparisons between reading heads are disallowed

Example: "comparison-free squaring"

$\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





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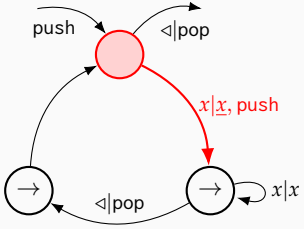

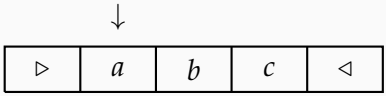

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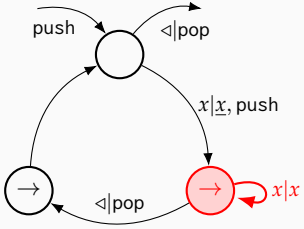

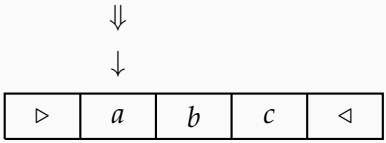

output $= \underline{a}$

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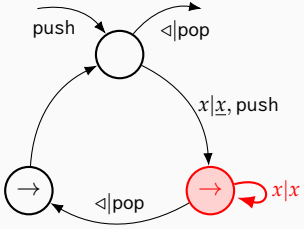

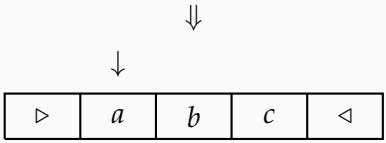

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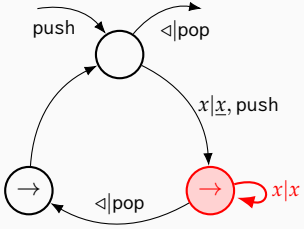

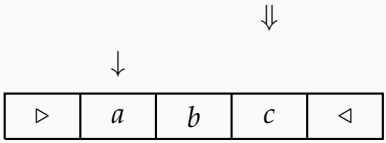

output $= \underline{a}ab$

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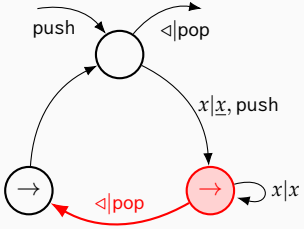

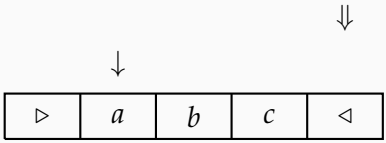

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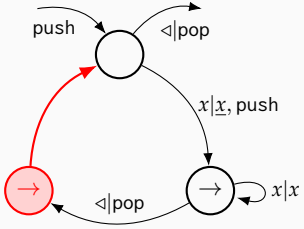

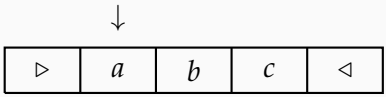

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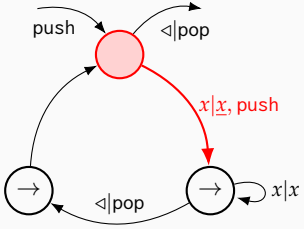

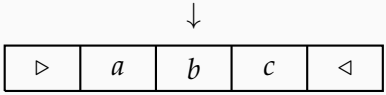

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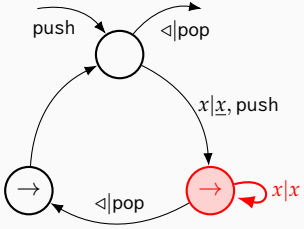

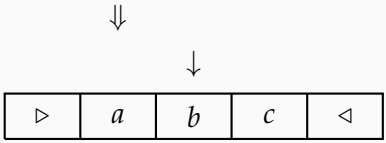

output $= \underline{a}abc\underline{b}$

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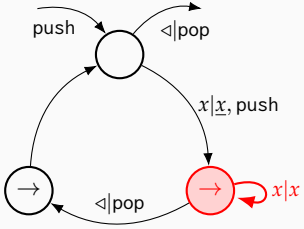

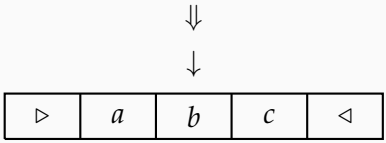

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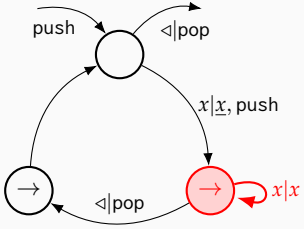

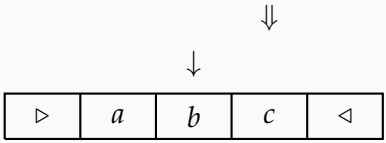

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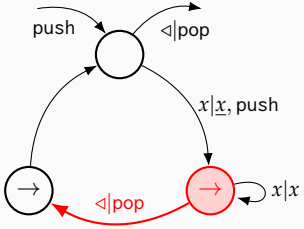

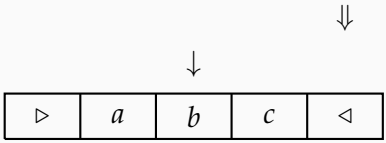

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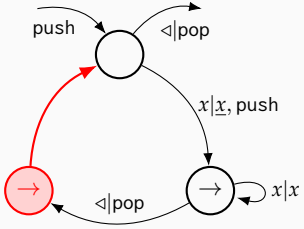

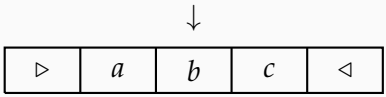

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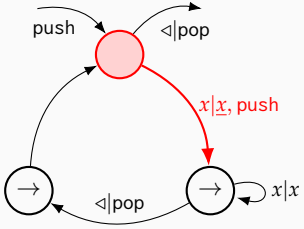

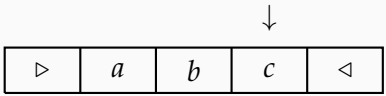

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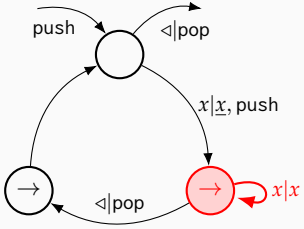

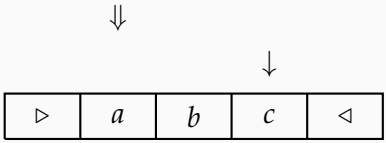

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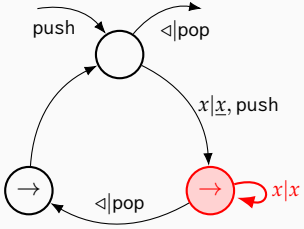

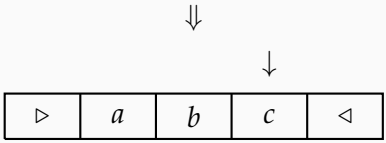

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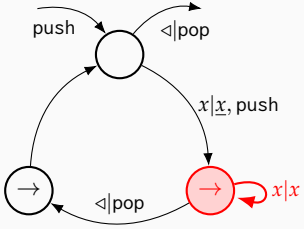

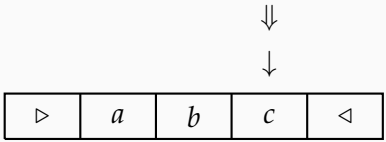

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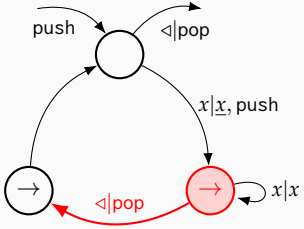

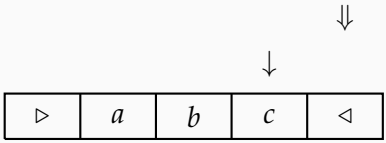

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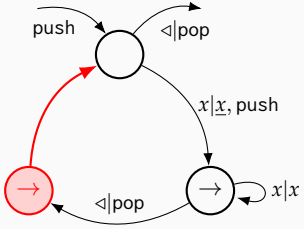

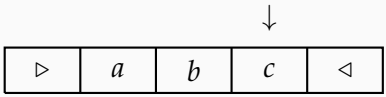

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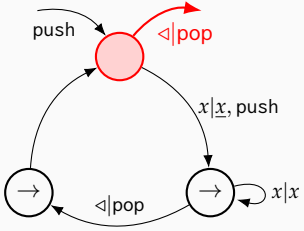

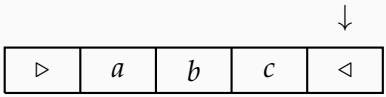

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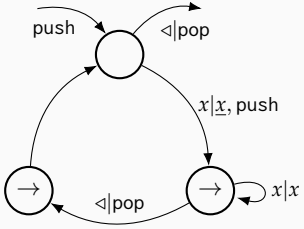
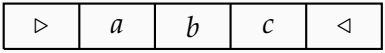
Blind transducers (a.k.a. "comparison-free pebble transducers")

Polyblind functions = computed by blind transducers

Pebble transducers where the comparisons between reading heads are disallowed

Example: "comparison-free squaring" $\texttt{cfsquaring}(abb) = \underline{a}abb\underline{b}abb\underline{b}abb$





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Contributions covered in this talk

- Alternative characterizations → "canonicity" / "naturality" claim
  (our original motivation: λ-calculus stuff, actually)
- Separation results (along the way: technical results such as a *pebble minimization* theorem)
- Discussion of special cases: unary inputs, integer sequences, aperiodicity

Alternative characterization (1/2): composition by substitutions

Definition (Composition by substitutions)

Let Γ, Σ and I be finite alphabets and $f : \Gamma^* \to I^*$, $g_i : \Gamma^* \to \Sigma^*$ and $w \in \Gamma^*$.
Define $\mathrm{CbS}(f, (g_i)_{i\in I})(w)$ so that, if $f(w) = i_1 \dots i_k$, then

$$\mathrm{CbS}(f, (g_i)_{i\in I})(w) = g_{i_1}(w) \dots g_{i_k}(w)$$

E.g. for `cfsquaring`, we take $f : \Sigma^* \to (\Sigma \cup \{X\})^*$, $g_X, g_a : \Sigma^* \to (\Sigma \cup \underline{\Sigma})^*$ (for $a \in \Sigma$) so that

$$f(abc) = aXbXcX \qquad g_a(w) = \underline{a} \quad \text{and} \quad g_X(w) = w$$

- Note: both polyblind and polyregular functions are closed under CbS

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Alternative definition of polyblind functions: smallest class such that...

- Every regular function is polyblind
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- Straightforward refomulation of blind transducers; more convenient to manipulate formally
- Number of pebbles = nesting of the CbS operator + 1

Alternative characterization (2/2): composition and basic combinators

Polyblind functions can be characterized using a linear λ-calculus (again, see my PhD thesis)
$\longrightarrow$ Hints at the following non-trivial theorem

(reproven with automata-theoretic tools in our ICALP'21 paper with no references to the λ-calculus)

Closure under composition

If $f : \Sigma^* \to \Gamma^*$ and $g : \Gamma^* \to \Delta^*$ are both polyblind, so is $g \circ f$.

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Leads to a combinator-based definition:

Alternative definition of polyblind functions

Least class containing the regular functions, `cfsquaring` and closed under composition.

- Analogous to the case of general polyregular functions

  `cfsquaring` replaced by "squaring with underlining" in the above $\to$ all polyregular functions

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- Note: regular functions can also themselves be obtained by composing:
  - sequential functions (that can themselves be decomposed by [Krohn & Rhodes 1965]);
  - $\texttt{mapReverse}_\Sigma, \texttt{mapDuplicate}_\Sigma : (\Sigma \cup \{\#\})^* \to (\Sigma \cup \{\#\})^*$ for $\# \notin \Sigma$

  (a theorem of Bojańczyk et al., see e.g. [Bojańczyk & Stefański 2020])

Separation results (or: what about counterexamples?)

Theorem

The function $f : a^n \in \{a\}^* \mapsto a\#aa\# \dots \#a^n$
is polyregular but not polyblind.

Corollary

"Squaring with underlining" is not polyblind.

Theorem

$g : a^{n_1}\# \dots \#a^{n_k} \in \{a, \#\}^* \mapsto a^{n_1 \times n_1}\# \dots \#a^{n_k \times n_k}$
is polyregular but not polyblind.

Note: stronger result in [Douéneau-Tabot 2021] (cf. next talk) on $a^{n_1}\# \dots \#a^{n_k} \mapsto a^{n_1 \times n_1 + \dots + n_k \times n_k}$

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f is also an **HDT0L transduction**

($\iff$ computable by a marble transducer / copyful streaming string transducer; *more in Gaëtan Douéneau-Tabot's talk next*)

$\longrightarrow$ HDT0L $\not\subset$ polyblind; conversely:

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$(w \in \Gamma^* \mapsto w^{|w|}) \in$ *polyblind* $\setminus$ *HDT0L for* $|\Gamma| \geq 2$.

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Definition

For $h : \Gamma^* \to \Sigma^*$, $w_1, \dots, w_n \in \Gamma^*$ with $\# \notin \Gamma$, $\mathbf{map}(h)(w_1\# \dots \#w_n) = f(w_1)\# \dots \#f(w_n)$.

Since $g = \mathbf{map}(w \mapsto w^{|w|})$:

Corollary

Polyblind functions are not closed under **map**, *unlike regular and polyregular functions.*

$\rightarrow$ obstruction to characterizing polyblind fn by list-processing functional programs à la regular/polynomial list functions

Separation proof for “map unary square” + blind pebble minimization

Theorem

$g(a^{n_1}\# \dots \# a^{n_k}) = a^{n_1 \times n_1}\# \dots \# a^{n_k \times n_k}$
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Proof by contradiction: assume g is polyblind.
First, $|g(w)| = O(|w|^2)$ *therefore* g is computed by some **2**-head blind transducer.

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If f is polyblind and $|f(w)| = O(|w|^k)$ then some blind transducer with k heads *computes f.*

More on this kind of result in Gaëtan's talk...

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(Oversimplified) idea: consider $g(w)$ where

$$w = \underbrace{aa \dots a\# \dots \#aa \dots a}_{|I| \text{ times } \# \text{ i.e. } |I|+1 \text{ blocks of } a\text{s}}$$

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"pump" p-th factor $aa \dots a$ $(1 \le p \le |I| + 1)$ in w
$\to$ p-th factor in $g(w)$ grows as $\Theta(n^2)$
$\to$ $\exists$ corresponding $j[p] \in I$ such that
$|f(w)|_{j[p]}$ and $|h_{j[p]}(w)|$ grow as $\Theta(n)$

Pigeonhole principle: $j[p] = j[q]$ for some $p \neq q$
$\to$ "pumping" p-th factor of w makes q-th factor of $g(w)$ grow, contradiction

Separation proof idea continued: unary inputs

Theorem

$f(a^n) = a\#aa\#\ldots\#a^n$ *is* not *polyblind.*

Observation: $f(a^n)$ has the n maximal a-factors

$$a \quad aa \quad \ldots \quad a^n$$

Lemma

For any polyblind $g : \{a\}^* \to \Sigma^*$, there are $O(1)$ possible lengths *for maximal* a*-factors in* $g(a^n)$.

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Smallest subclass of $(\Sigma^*)^{\mathbb{N}}$

- Containing the constant sequences $\alpha_n = w$
- Closed under concatenation $\alpha_n = \beta_n \cdot \gamma_n$
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$\to$ "ultimately periodic combinations" (u.p.c.)

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  Proof idea: find an idempotent in a suitable transition monoid of your favorite machine model for reg fn
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- Work-in-progress: similar characterization of polyregular word sequences

First-order comparison-free polyregular functions (unpublished)

Poly-pumping sequences: generated from constants by concat and "iteration" $\alpha_n = (\beta_n)^n$.

Theorem

"ultimately periodic combinations" of poly-pumping sequences = all polyblind sequences

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FO-regular functions: many equivalent def.

- logic: replace MSO by FO
- 2DFT/SST w/ aperiodic monoid of behaviors
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- regexp-like [Dartois, Gastin & Krishna 2021]
- more self-advertizing:
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Integer sequences (unpublished)

Previously: unary *inputs*. Gaëtan's talk (next):
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1. HDT0L ($\Leftrightarrow$ linear recurrence) s.t. $f(n) = n^{O(1)}$
2. ultimately periodic combinations of $\mathbb{N}[X]$
   (unary poly-pumping sequences = polynomials in $\mathbb{N}[X]$)
3. polyblind integer sequences

$\to$ single canonical automata-theoretic class of polynomial growth integer sequences?

polyregular $\Leftrightarrow$ (1) $\Leftrightarrow$ (2) holds for arbitrary input and unary ouptut [Douéneau–Tabot 2021]; the rest is elementary

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Theorem

Let $f : \mathbb{N} \to \mathbb{N}$. Then f is FO-polyblind $\iff$
$\exists P \in \mathbb{Z}[X] : \forall n$ *except finitely many,* $f(n) = P(n)$

Thus, the following is not FO-polyblind for $k \geq 2$:

$$b_k(n) = \binom{n}{k} \qquad \text{e.g. } b_2(n) = \frac{n(n-1)}{2}$$

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and FO-polyreg (think of $a^n \mapsto a\#aa\#\ldots\#a^n$)
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$$\forall P \in \mathbb{Q}[X],\ P(X+a) = \sum_{i=0}^{\deg P} \Delta^k(P)(a) \binom{X}{k}$$

$\to$ all polynomials P s.t. $P(\mathbb{N}) \subseteq \mathbb{N}$ are FO-polyreg

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Let $f : \mathbb{N} \to \mathbb{N}$. Then f is FO-polyregular $\iff$ $\exists P \in \mathbb{Q}[X] : \forall n$ *except finitely many,* $f(n) = P(n)$

Not a logical characterization of FO-polyblind functions

Let $\mathfrak{M} : \{\text{words}\} \to \{\text{finite models}\}$ be as usual.
For $\mathfrak{U} = (U, R, \ldots)$, let $\mathfrak{U}^k = (U^k, R_1, \ldots, R_k, \ldots)$ where $R_i(x_1, \ldots, x_m) :\Leftrightarrow R(\pi_i(x_1), \ldots, \pi_i(x_m))$.

Reasonable-sounding conjecture

f is FO-polyblind if and only there exist $k \in \mathbb{N}$ and a FO transduction φ such that

$$\forall w.\ \ \mathfrak{M}(f(w)) \ \simeq\ \varphi\left(\mathfrak{M}(w)^k\right)$$

Idea: in $\mathfrak{M}(w)^k$, one cannot compare different coordinates $\longrightarrow$ "comparison-free" condition

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- Defines a subclass of FO-polyregular functions, according to [Bojańczyk, Kiefer & Lhote 2019]
- Beware: naive replacement of FO with MSO $\to$ non reg-preserving
- FO-polyblind $\subsetneq$ this kind of FO-interpretation; separating example:

$$w_1 \# \ldots \# w_n \mapsto (w_1)^n \ldots (w_n)^n$$

  - it requires a k-pebble transducer for $k \geq 3$; apply the blind pebble minimization theorem
  - j.w.w. Mikołaj Bojańczyk, Gaëtan Douéneau-Tabot, Sandra Kiefer et Pierre Pradic

Conclusion

A new(?) class of string-to-string functions: *polyblind* (or *comparison-free polyregular*) functions.

Equivalent definitions

- by blind transducers
- inductively (composition by substitution)
- **as the composition closure of regular functions +** $\texttt{cfsquaring}(abc) = \underline{a}abc\underline{b}abc\underline{c}abc$
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Not as robust as hoped at first:
polyblind functions are *not* equivalent to

- "comparison-free" logical interpretations
- a restriction of polynomial list functions

$\longrightarrow$ are there other "interesting" subclasses of polyregular functions?