Comparison-free polyregular functions

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joint work with Pierre Pradic (University of Oxford)
and the fictional author Camille Noûs (`https://www.cogitamus.fr/camilleen.html`)

June 16th, 2021 — Automata Theory seminar, MIMUW, University of Warsaw

Background: two-way transducers → regular functions

Two-way (deterministic finite) transducer (2DFT):
finite state + bidirectional reading head
+ output produced from left to right

Example: $w \in \{a, b\}^* \mapsto a^{|w|} \cdot \mathtt{reverse}(w)$

$\triangleright$	a	b	a	c	$\triangleleft$

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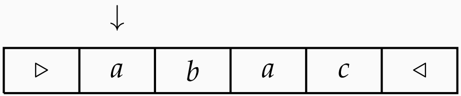

state $= 1$ output $=$

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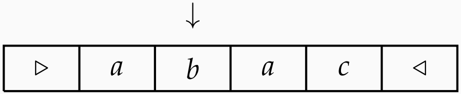

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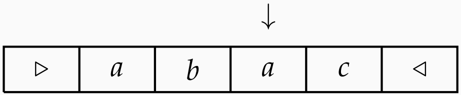

state $= 1$ output $= aa$

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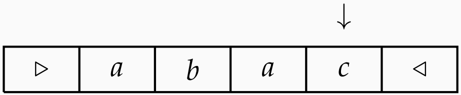

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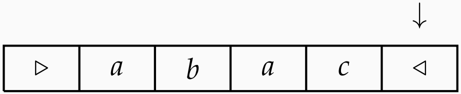

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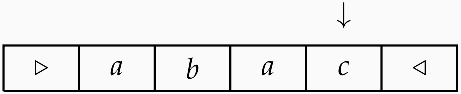

state $= 2$ output $= aaaa$

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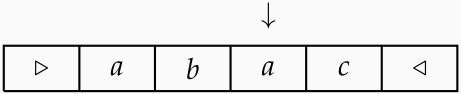

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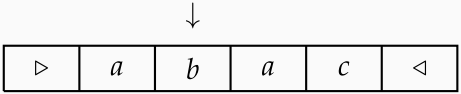

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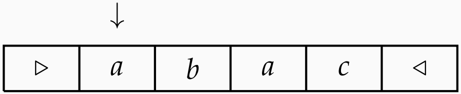

state $= 2$ output $= aaaacab$

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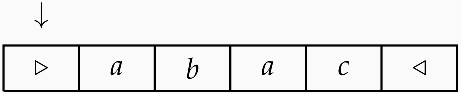

state = 2 output = *aaaacaba*

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state = $\texttt{end}$ output = $aaaacaba$

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state = `end` output = $aaaacaba$

Can also change direction in middle of input,
e.g. $\mathtt{mapReverse}(abc\#bac\#ca) = cba\#cab\#ac$

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- linear growth: $|f(w)| = O(|w|)$
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Some alternative characterizations:

- via Monadic Second-Order logic
- copyless streaming string transducers
- various functional programming or regexp-like (declarative) formalisms
- (self-advertizing) linear λ-calculus (*Implicit automata in typed λ-calculi*)

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k-pebble transducers ($k \in \mathbb{N}_{\geq 1}$):
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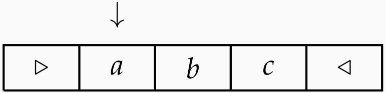

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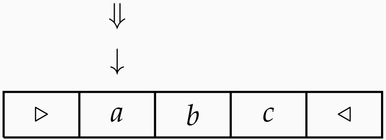

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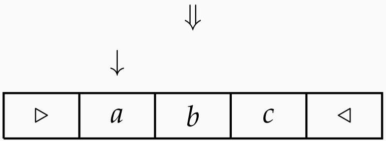

output = $\underline{a}$

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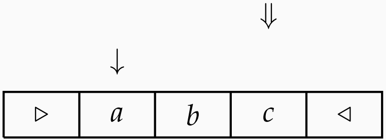

output $= \underline{a}b$

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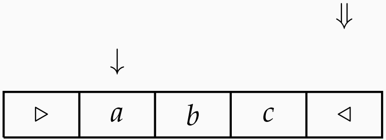

output = $\underline{a}bc$

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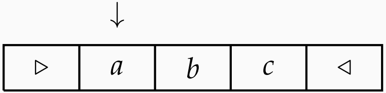

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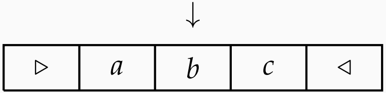

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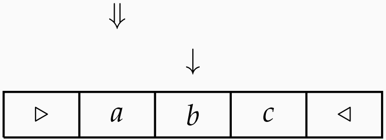

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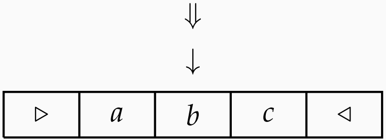

output $= \underline{a}bca$

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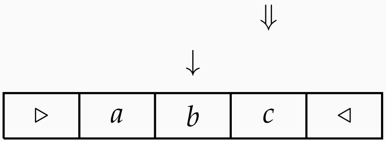

output $= \underline{a}bca\underline{b}$

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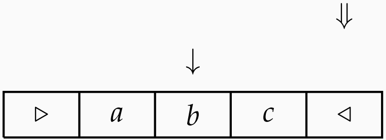

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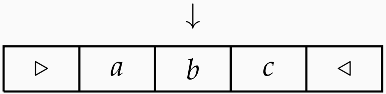

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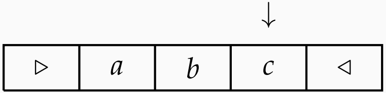

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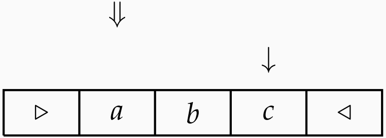

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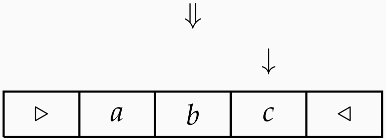

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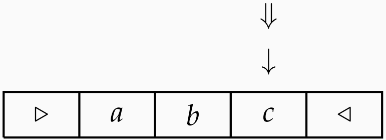

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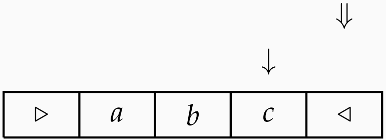

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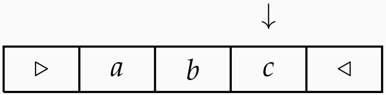

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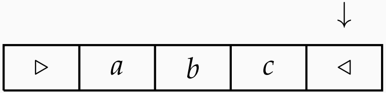

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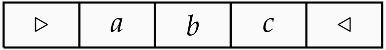

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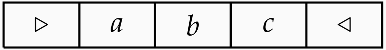

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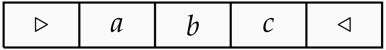
Recent characterizations [Bojańczyk 2018; Bojańczyk, Kiefer & Lhote 2019]:

- multidimensional MSO interpretations
- imperative nested loop programs
- simply typed λ-calculus + list manipulation primitives
- composition closure of [regular functions ∪ "squaring with underlining"]

Background: pebble transducers → polyregular functions

k-pebble transducers ($k \in \mathbb{N}_{\geq 1}$):
finite state + *stack of at most k two-way heads*

Ex: “squaring with underlining” ($k = 2$)



output $= \underline{a}bca\underline{b}cab\underline{c}$

- Operations on reading head stack: move topmost / push / pop
- Transitions can *compare* head positions
  - **Our paper: if they can’t, then what?**

Properties of *polyregular* functions:

- *polynomial* growth: $|f(w)| = O(|w|^k)$
- closed under composition
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Introducing comparison-free polyregular functions

(new definitions/results start here)

Comparison-free k-pebble transducers:

finite state + stack of at most k two-way heads
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$\texttt{cfsquaring}(abc) = \underline{a}abc\underline{b}abc\underline{c}abc$

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- composition closure of reg fn $\cup$ `cfsquaring`
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  hard part: cfpolyreg fn are closed under comp
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- *conjecture*: some kind of logical transduction

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Inductive reformulation of the cf pebble def:
the smallest class such that

- every regular function is cfpolyreg
- if f is regular and g_i is cfpolyreg $\forall i \in I$ then $\mathrm{CbS}(f, (g_i)_{i\in I})$ is cfpolyreg

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Note: both cfpolyreg and polyregular
functions are closed under CbS

Separation results

What about *counter*examples?

Theorem

The function $f : a^n \in \{a\}^* \mapsto a\#aa\# \dots \#a^n$ *is polyregular but not comparison-free.*

Corollary: “squaring with underlining” is not CF.

Theorem

$g : a^{n_1}\# \dots \#a^{n_k} \in \{a, \#\}^* \mapsto a^{n_1 \times n_1}\# \dots \#a^{n_k \times n_k}$ *is polyregular but not comparison-free.*

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$w \in \Gamma^* \mapsto w^{|w|}$ *is comparison-free polyregular, but when* $|\Gamma| \geq 2$*, it is not HDT0L.*

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→ obstruction to characterizing cfpolyreg fn by list-processing functional programs (à la [Bojańczyk, Daviaud & Krishna 2018])

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Separation proof for “map unary square” + pebble minimization

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If f is cfpolyreg and $|f(w)| = O(|w|^k)$ then some comparison-free k-pebble transducer computes f.

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"pump" p-th factor $aa\dots a$ $(1 \le p \le |I|+1)$ in w
→ p-th factor in $g(w)$ grows as $\Theta(n^2)$
→ $\exists$ corresponding $j[p] \in I$ such that $|f(w)|_{j[p]}$ and $|h_{j[p]}(w)|$ grow as $\Theta(n)$

Pigeonhole principle: $j[p] = j[q]$ for some $p \neq q$
→ "pumping" p-th factor of w makes q-th factor of $g(w)$ grow, contradiction

Separation proofs continued + unary inputs

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Observation: $f(a^n)$ has the n maximal a-factors

$$a \quad aa \quad \dots \quad a^n$$

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Smallest subclass of $(\Sigma^*)^{\mathbb{N}}$

- containing the constant sequences $\alpha_n = w$
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$f : \{a\}^* \to \Sigma^*$ *is comparison-free polyregular if and only if* $\exists p \in \mathbb{N}$ *such that* $(f(a^{(n+1)p+m}))_{n\in\mathbb{N}}$ *is poly-pumping for every* $m < p$.

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- Regular word sequences are u.p.c. of pumping sequences $(u_0(v_1)^n \dots (v_l)^n u_l)_{n\in\mathbb{N}}$ [Choffrut 2017]

  Proof idea: find an idempotent in a suitable transition monoid of your favorite machine model for reg fn
- Proof for general cfpolyreg sequences: induction on the CbS-based definition

First-order comparison-free polyregular functions (not in the paper)

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Previously: unary *inputs*. [Douéneau-Tabot 2021]:
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3. *f is an ultimately periodic combination of $\mathbb{N}[X]$*

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polyreg ⇔ (1) ⇔ (2) holds for arbitrary input and unary ouptut [DT21]; the rest is elementary

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Theorem

f is FO-cfpolyreg $\Leftrightarrow \exists p \in \mathbb{N} : f(X + p) \in \mathbb{N}[X] \Leftrightarrow$ $\exists P \in \mathbb{Z}[X] : \forall n$ except finitely many, $f(n) = P(n)$

Thus, the following is not FO-cfpolyreg for $k \geq 2$:

$$b_k(n) = \binom{n}{k} \qquad \text{e.g. } b_2(n) = \frac{n(n-1)}{2}$$

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Previously: unary *inputs*. [Douéneau-Tabot 2021]: unary *outputs*. Let's consider both: let $f : \mathbb{N} \to \mathbb{N}$.

Theorem

f is polyregular $\iff$ any of those:

1. *f is HDT0L ($\Leftrightarrow$ linear recurrence) & $f(n) = n^{O(1)}$*
2. *f is computable by a "marble bimachine"*
3. *f is an ultimately periodic combination of $\mathbb{N}[X]$*
   (*unary poly-pumping sequences = polynomials in* $\mathbb{N}[X]$)
4. *f is comparison-free polyregular*

→ single canonical automata-theoretic class of polynomial growth integer sequences?

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f is FO-cfpolyreg $\Leftrightarrow \exists p \in \mathbb{N} : f(X+p) \in \mathbb{N}[X] \Leftrightarrow$ $\exists P \in \mathbb{Z}[X] : \forall n$ except finitely many, $f(n) = P(n)$

Thus, the following is not FO-cfpolyreg for $k \geq 2$:

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Yet all b_k are both cfpolyreg (divide by $k!$) and FO-polyreg (think of $a^n \mapsto a\#aa\#\ldots\#a^n$)
→ **FO-cfpolyreg ≠ (FO-polyreg ∩ cfpolyreg)**

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$$\forall P \in \mathbb{Q}[X],\ P(X+a) = \sum_{i=0}^{\deg P} \Delta^k(P)(a) \binom{X}{k}$$

→ all polynomials P s.t. $P(\mathbb{N}) \subseteq \mathbb{N}$ are FO-polyreg (*conjecture*: some kind of converse holds)

Conclusion

A new(?) class of string-to-string functions: *comparison-free polyregular functions*.

Equivalent definitions

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- inductively (composition by substitution)
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Conjecture (does not work naively with MSO!)

f is *first-order* comparison-free polyregular

$\iff$ there exist $k \in \mathbb{N}$ & a FO transduction φ such that $\forall w,\ \mathfrak{M}(f(w)) \simeq \varphi\left(\mathfrak{M}(w)^k\right)$.

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(Thanks for your attention! Questions?)

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