A complexity gap between pomset logic and system BV

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The big picture

Reminder from Christian's talk: *pomset logic* conservatively extends MLL+Mix with '$\triangleleft$'

$$A, B ::= \mathbf{I} \mid a \mid a^{\perp} \mid A \otimes B \mid A \parr B \mid A \triangleleft B$$

non-commutativity: $A \triangleleft B \not\equiv B \triangleleft A$ self-duality: $(A \triangleleft B)^{\perp} = A^{\perp} \triangleleft B^{\perp}$ (*not* $B^{\perp} \triangleleft A^{\perp}$)

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Guglielmi 2007, *A System of Interaction and Structure* (emphasis mine):

> **It is still open whether the logic in this paper, called *BV*, is the same as pomset logic. We conjecture that it is actually the same logic**, *but one crucial step is still missing, at the time of this writing, in the equivalence proof. This paper is the first in a planned series of 3 papers dedicated to BV.* [...] *In the 3rd part, some of my colleagues will hopefully show the equivalence of BV and pomset logic,* [...]

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It is known that BV provability implies pomset provability. **We show that the converse is false!** Explicit counterexample with 8 variables, each with 1 positive and 1 negative occurrence

Proof nets vs sequent calculus vs deep inference

Pomset logic is defined through *proof nets* with a "geometric" correctness criterion
No complete *deductive* proof system known until Slavnov's sequent calculus (2019)
(deductive = inference rules + derivation trees)

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So if pomset logic and BV had been equivalent – *we show that they aren't!* – this would have:

- provided a deductive system for pomset logic (deep inference is deductive);
- explained the common difficulty.

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> *This paper* [that is, Guglielmi 2007] *is the first in a planned series of 3 papers dedicated to BV.* [...]
> *In the 2nd part, Alwen Tiu will show why BV cannot be defined in any sequent system.*

There is indeed an obstruction that rules out traditional sequent calculi and "shallow systems"
more generally (*A System of Interaction and Structure II: The Need for Deep Inference*, 2006)
but it doesn't apply to e.g. Slavnov's "decorated sequents"

A glance at deep inference + complexity of provability

Deep inference = unary rules applied to subformulas of arbitrary depth:

$$\text{inference rule } \frac{A}{B} \qquad \rightsquigarrow \qquad \text{instances } \frac{S[A]}{S[B]} \text{ for any context } S$$

(compare with rewriting systems, or functoriality in categorical logic)

$$\text{e.g.} \qquad \frac{A \parr (B \otimes (C \parr D))}{A \parr ((B \otimes C) \parr D)}$$

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I won't list the rules of system BV (they're related to the cograph rewriting in Christian's talk)

Important: *proofs of A are of length $O(|A|^2)$*, therefore **provability in BV is a NP problem**

(actually NP-complete [Kahramanoğulları 2008])

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Theorem (the actual main topic of the talk)

Provability in pomset logic is Σ_2^{p}-complete.

(2nd level of the polynomial hierarchy, *strictly harder than* NP unless NP = coNP)

Complexity of proof net correctness

Theorem

Provability in pomset logic is Σ_2^{p}-complete.

Along the way, we proved a simpler result with the same tools:

Theorem

Proof net correctness (given a proof structure, is it a correct proof net?) in pomset logic is coNP-complete.

The rest of the talk proves this!
I will assume that you know about MLL+Mix proof nets

Important ingredient: *perfect matchings*, generalized to *directed* graphs
+ inspiration from the literature on *edge-colored graphs*
(Gourvès et al. 2013, *Complexity of trails, paths and circuits in arc-colored digraphs*)

Perfect matchings

Definition

A *perfect matching* is a set of edges in a graph such that each vertex is incident to exactly one edge in the matching.

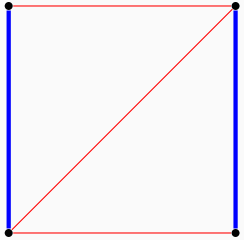
A classical topic in combinatorics!

Definition

An *æ-cycle* is a cycle which

- *alternates* between edges inside and outside the matching
- is *elementary*: has no vertex repetitions

Remark: $\exists$ æ-cycle $\Leftrightarrow$ the perfect matching is not *unique*
(Often just called "alternating cycles", elementary by default)



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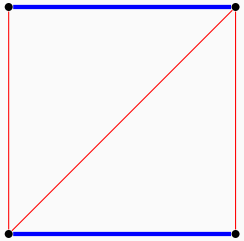
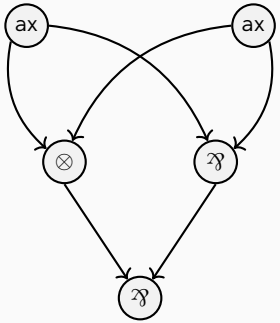
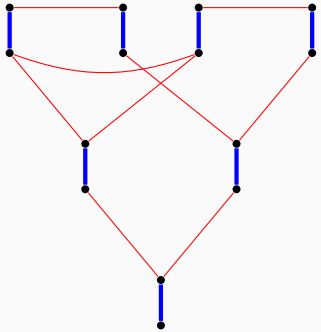


Perfect matchings for MLL+Mix proof nets

An old translation proof structures → graphs with perfect matchings by Christian Retoré
(*beware*: not to be confused with cographs & chordless æ-cycles!)





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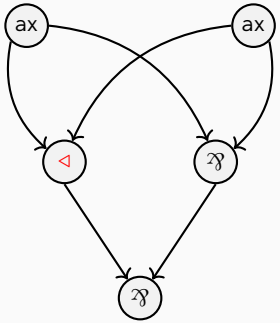

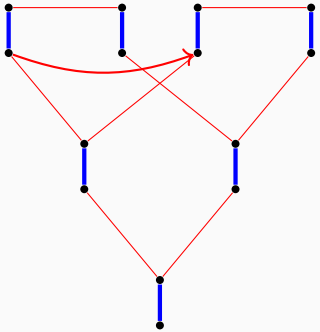

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Pomset logic: add *directed* edges to handle non-commutativity

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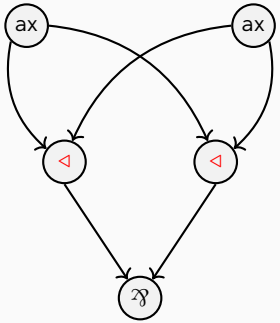

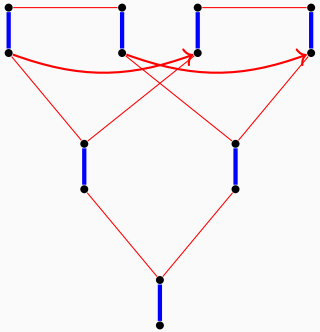

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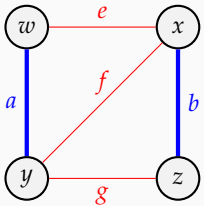
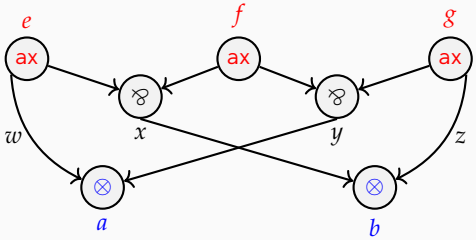
Reduction perfect matchings → proof structures

The converse direction (from my paper *Unique perfect matchings and proof nets*, 2018):





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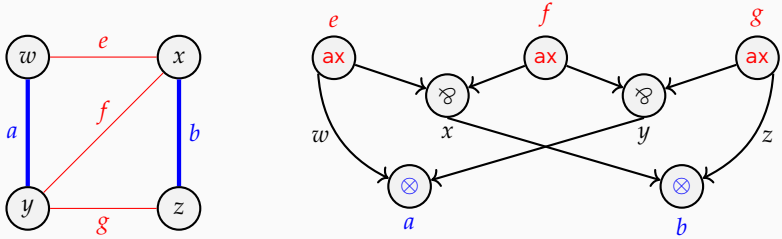


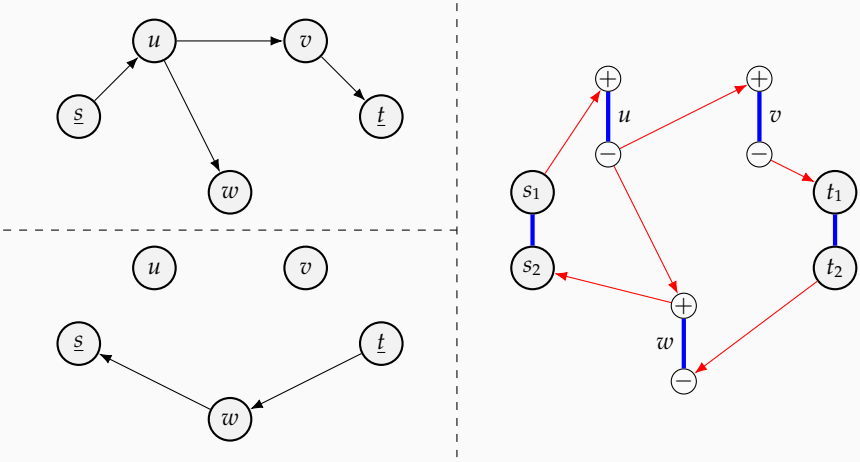
Extends to perfect matchings in general *directed* graphs → *pomset logic* proof structures, using the non-commutative ◁ to build a "directed ax" gadget. We therefore have:

Lemma

There is a polynomial-time reduction from ∃ of directed æ-cycles to pomset logic proof net incorrectness.

Goal: show that finding directed æ-cycles is NP-hard

Reduction from another graph-theoretic problem ("elementary round-trip")



æ-cycle on the right $\iff$ go from s to t in top graph, then back from t to s in bottom graph, *without visiting the same vertex twice* (elementary $\in$ æ)

Reduction CNF-SAT $\rightarrow$ elementary round-trip

Consider the SAT instance

$$(x \vee y \vee z) \wedge (\neg x \vee y) \wedge (\neg y \vee \neg z)$$

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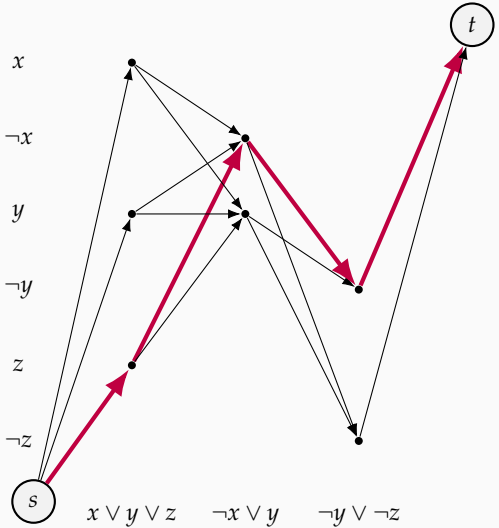

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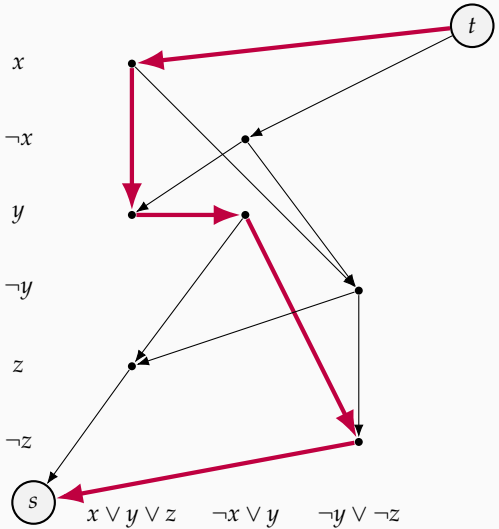

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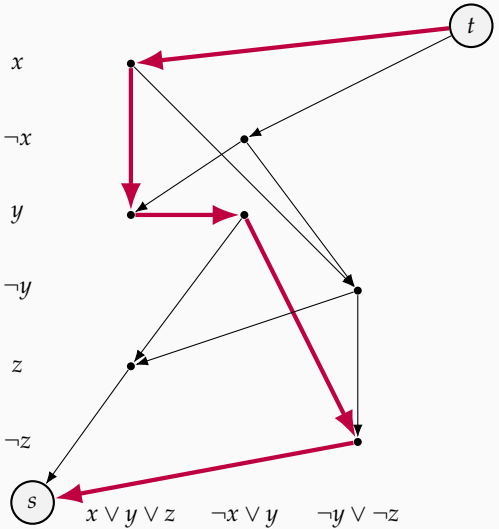
Non-intersecting pair (cf. previous slide)
= satisfying assignment



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Provability is a strictly harder problem for pomset logic than for BV unless NP = coNP:

- it is NP-complete for system BV [Kahramanoğulları 2008]
- and Σ_2^{P}-complete for pomset logic (new!)
  - related result: proof net correctness is coNP-complete in pomset logic (also new!)

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Unconditional refutation of the conjectured equivalence of pomset logic and BV:

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