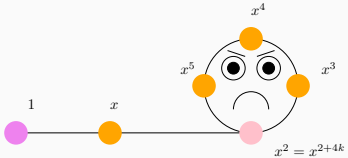

Implicit automata in typed λ-calculi I: Aperiodicity in a non-commutative logic

Nguyễn Lê Thành Dũng (Laboratoire d'informatique de Paris Nord)
and Pierre Pradic (University of Oxford)

ICALP 2020 (pre-recorded video talk)

Background: characterizations of regular languages

Regular expressions

e.g. $L = b^*(ab^*ab^*)^*$

"even number of 'a's"

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Broader project: type-theoretic implicit complexity for automata theory

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More background: Church encodings in the simply typed λ-calculus

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$$f\,t \;\approx\; f(t) \qquad \lambda x.\, t \;\approx\; (x \mapsto t)$$

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Church encoding of natural numbers, untyped: 1930s
Arbitrary free algebras, with types: Böhm & Berarducci 1985

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From star-free languages to non-commutativity via aperiodic monoids

Definition

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- without repetition star L^*

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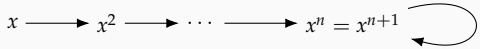

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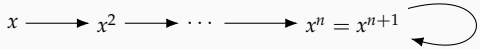

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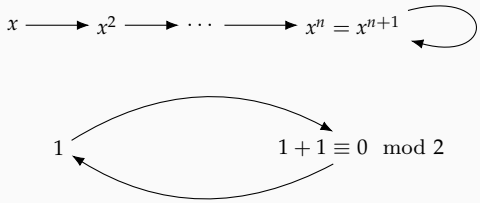

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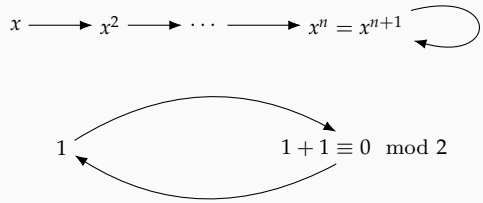

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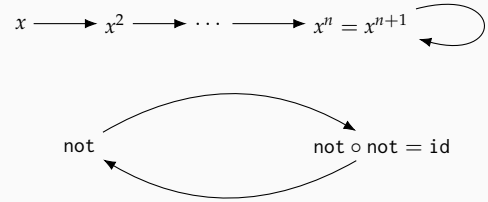

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$\texttt{not}^{n+1} \neq \texttt{not}^{n}$

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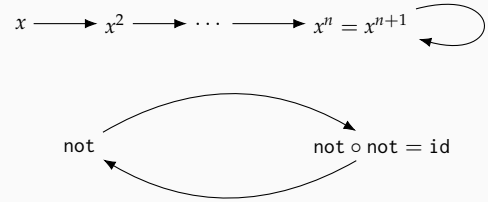

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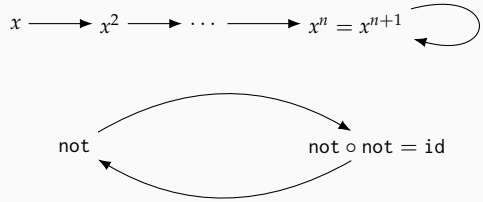

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From star-free languages to non-commutativity via aperiodic monoids

Definition

Star-free language: definable by regexp

- without repetition star L^*
- with *complementation*: $L^c = \Sigma^* \setminus L$

$(ab)^* = (b\varnothing^c | \varnothing^c a | \varnothing^c aa \varnothing^c | \varnothing^c bb \varnothing^c)^c \quad (\Sigma = \{a, b\})$

However $(aa)^*$ does not have a star-free regexp...
its "minimal monoid" is $\mathbb{Z}/(2)$: *not aperiodic*

Definition

A monoid M is *aperiodic* when for any $x \in M$ there exists an exponent $n \in \mathbb{N}$ such that $x^n = x^{n+1}$.

Theorem (Schützenberger 1965)

$L \subseteq \Sigma^*$ *is* star-free $\iff$ $L = \varphi^{-1}(P)$ *with*

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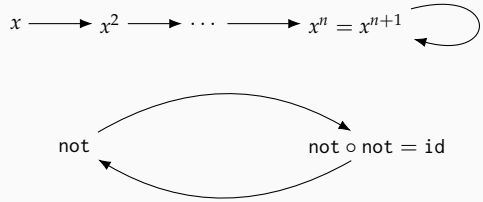

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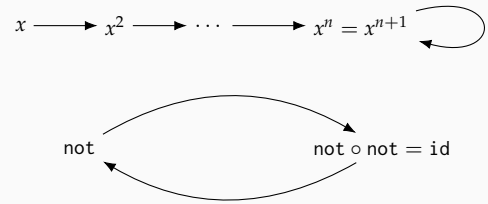

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Idea: make the order of arguments matter

"a function $\lambda b.\ \lambda x.\ \lambda y.\ (\ldots)$ should not use x after y, since $\lambda x.$ occurs before λy."
$\longrightarrow$ *non-commutative* λ*-calculus*

Designing a non-commutative affine λ-calculus

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The language $L \subseteq \Sigma^*$ is *star-free*
$\iff$
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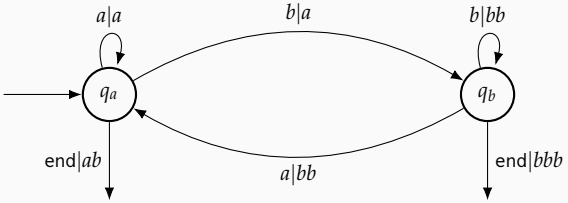

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The language $L \subseteq \Sigma^*$ *is* star-free

$\Updownarrow$

L is defined by some $t : \mathtt{Str}_\Sigma\{o := A\} \multimap \mathtt{Bool}$
in our non-commutative affine λ*-calculus*
with A purely affine, *i.e. containing no* '$\to$'

($\Uparrow$) Key lemma: for purely affine A, the monoid
$\{t \mid t : A \multimap A\}/=_\beta$ is finite & aperiodic
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($\Downarrow$) encoding of *aperiodic sequential transducers*
at type $\mathtt{Str}_\Gamma\{o := A\} \multimap \mathtt{Str}_\Sigma$
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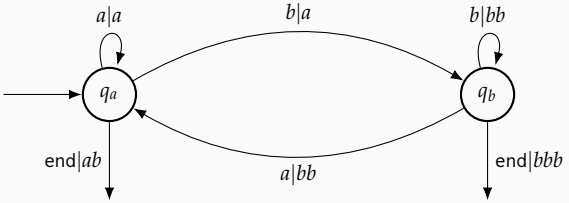
via "prime decomposition"
[Krohn & Rhodes 1965]
"prime" transducers have 2 states



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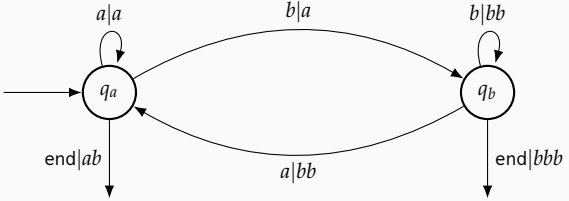

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$\mathtt{Str}_\Gamma\{o := A\} \multimap \mathtt{Str}_\Sigma \iff$ aperiodic *regular* functions
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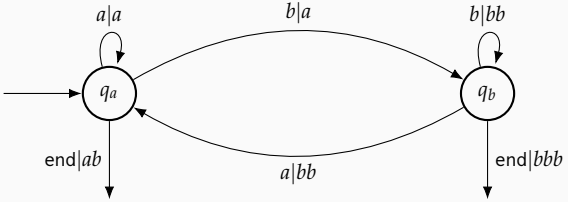
Commutative affine types $\implies$ *regular instead of star-free*



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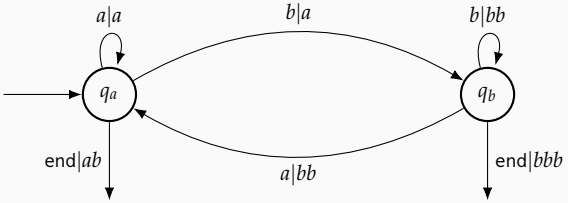
Commutative affine types $\implies$ *regular instead of star-free*

$\longrightarrow$ non-commutative types are computationally meaningful



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`https://hal.archives-ouvertes.fr/hal-02476219`

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