Implicit automata in typed λ-calculi

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joint work with Pierre Pradic (University of Oxford)

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The idea

Suppose the programs of type T in a programming language $\mathcal{P}$ all compute *languages*, something like $T = \texttt{String} \to \texttt{Bool}$.
(Or *functions* $\texttt{String} \to \texttt{String}$.)

What class of languages? Depends on $\mathcal{P}$ and T.
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My research project w/ Pierre Pradic:
The same thing for **automata theory** instead of complexity

Previous work

Template for a theorem

The languages/functions computed by programs of type T in $\mathcal{P}$ are the same as those computed by the class $\mathcal{A}$ of automata.

	$\mathcal{P}$	$\mathcal{A}$
[1]	simply typed λ-calculus	regular languages

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Defining languages in the simply typed λ-calculus

We use *Church encodings* [Böhm & Berarducci 1985]:

Binary strings: think of `fold_right` on a list of characters

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Theorem (Hillebrand & Kanellakis 1996)

This can define all the regular languages*, and only those.*

Regular languages

Many equivalent definitions:

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Theorem (classical)

A language $L \subseteq \Sigma^*$ *is* regular *if and only if there exists a* monoid morphism $\varphi : \Sigma^* \to M$ *to a* finite monoid M *and a subset* $P \subseteq M$ *such that* $L = \varphi^{-1}(P) = \{w \in \Sigma^* \mid \varphi(w) \in P\}$.

Σ: finite alphabet, Σ^*: words over Σ

monoid structure: for $v, w \in \Sigma^*$, $v \cdot w$ = concatenation

morphism: for $w \in \Sigma^*$ with n letters, $\varphi(w) = \varphi(w[0]) \dots \varphi(w[n])$

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- *algebraic* definition below
- simply typed λ-calculus + Church encodings!

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λ-definable languages are regular

Theorem (Hillebrand & Kanellakis, LICS'96)

For any type A & any simply typed λ-term $t : \mathsf{Str}_{\{0,1\}}[A] \to \mathsf{Bool}[o]$, the language $\{w \in \{0,1\}^* \mid t\,\overline{w} =_\beta \mathtt{true}\}$ is regular.

Proof sketch: by semantic evaluation.

Use the standard semantics in *finite sets*: $[\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$, so that $[\![\mathsf{Str}[A]]\!]$ is finite.

$[\![\overline{w}]\!]$ determines whether $[\![t\,\overline{w}]\!] = [\![t]\!]\,([\![\overline{w}]\!]) = [\![\mathtt{true}]\!]$. When $\mathrm{Card}([\![o]\!]) \geq 2$, $[\![\mathtt{true}]\!] \neq [\![\mathtt{false}]\!]$, so this means $t\,\overline{w} =_\beta \mathtt{true}$.

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$[\![\mathsf{Str}[A]]\!]$ has a monoid structure $([\![\mathsf{concat}]\!], [\![\overline{\varepsilon}]\!])$ such that $w \in \{0,1\}^* \to [\![\overline{w}]\!] \in [\![\mathsf{Str}[A]]\!]$ is a monoid morphism. □

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Similar ideas in higher-order model checking, e.g. Grellois & Melliès

Our work

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$\mathcal{P}$	$\mathcal{A}$
non-commutative λ-calculus	star-free languages, aperiodic reg. fun.
(commutative) affine λ-calculus	regular functions

Star-free languages: ICALP 2020 paper.

Other items in preparation.

Star-free languages and aperiodicity

Star-free languages: regular expressions with *complementation* but *without repetition star*

e.g. for $\Sigma = \{a, b\}$, $(ab)^* = (b\varnothing^c|\varnothing^c a|\varnothing^c aa\varnothing^c|\varnothing^c bb\varnothing^c)^c$

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Theorem (Schützenberger 1965)

A language $L \subseteq \Sigma^*$ is star-free *if and only if there exists a monoid morphism $\varphi : \Sigma^* \to M$ to a finite and* aperiodic *monoid M and a subset $P \subseteq M$ such that $L = \varphi^{-1}(P) = \{w \in \Sigma^* \mid \varphi(w) \in P\}$.*

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Definition

A monoid M is *aperiodic* when for any $x \in M$ there exists an exponent $n \in \mathbb{N}$ such that $x^n = x^{n+1}$.

Morally, $(aa)^*$ involves the group $\mathbb{Z}/(2)$: not aperiodic

From aperiodicity to non-commutativity

How to enforce aperiodicity in a typed λ-calculus?
Embedding of $\mathbb{Z}/(2)$ via $\mathtt{not} : \mathsf{Bool} \to \mathsf{Bool}$ ($\mathtt{not} \circ \mathtt{not} =_\beta \mathtt{id}$)

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morally, $\mathtt{if}\ b\ \mathtt{then}\ x\ \mathtt{else}\ y \rightsquigarrow \mathtt{if}\ \mathrm{not}(b)\ \mathtt{then}\ y\ \mathtt{else}\ x$
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Idea: make the order of arguments matter

"a function $\lambda b.\ \lambda x.\ \lambda y.\ (\ldots)$ should first use b, then x, then y"
$\longrightarrow$ non-commutative type system

(Lambek 1958: non-commutative λ-calculus for linguistics)

Non-commutative types and linear logic

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(If "*exactly once*": non-commutative *linear* λ-calculus;
correspondence with *planar combinatorial maps* (N. Zeilberger))

Finally, our theorem

Our type system: a base type o + *two* function arrows:

- non-commutative affine functions: $\lambda^{\circ} x.\ t : A \multimap B$
- unrestricted functions: $\lambda^{\rightarrow} x.\ t : A \to B$

$$\mathsf{Str}_{\{\mathtt{0},\mathtt{1}\}}[A] = (A \multimap A) \to (A \multimap A) \to (A \multimap A)$$

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Theorem (N. & Pradic, ICALP 2020)

This typed λ-calculus can define all star-free languages*, and only those, with terms of type* $\mathsf{Str}_{\{\mathtt{0},\mathtt{1}\}}[A] \multimap \mathsf{Bool}[o]$ *where A is* purely affine *i.e. does not contain any unrestricted function arrow.*

(*A may vary depending on the language, as in Hillebrand & Kanellakis.*)

With commutative affine types, you'd get regular languages.

A non-commutative affine type system

Typing judgments $\Gamma \mid \Delta \vdash t : A$ for a *set* Γ and an **ordered list** Δ

$$\frac{}{\Gamma \uplus \{x : A\} \mid \varnothing \vdash x : A} \qquad \frac{}{\Gamma \mid x : A \vdash x : A}$$

$$\frac{\Gamma \mid \Delta \vdash t : A \to B \qquad \Gamma \mid \varnothing \vdash u : A}{\Gamma \mid \Delta \vdash t\,u : B} \qquad \frac{\Gamma \uplus \{x : A\} \mid \Delta \vdash t : B}{\Gamma \mid \Delta \vdash \lambda^{\to} x.\, t : A \to B}$$

$$\frac{\Gamma \mid \Delta \vdash t : A \multimap B \qquad \Gamma \mid \Delta' \vdash u : A}{\Gamma \mid \Delta \cdot \Delta' \vdash t\,u : B} \qquad \frac{\Gamma \mid \Delta \cdot (x : A) \vdash t : B}{\Gamma \mid \Delta \vdash \lambda^{\circ} x.\, t : A \multimap B}$$

$\approx$ Polakow & Pfenning's Non-Commutative Intuitionistic Linear Logic

$$\frac{\Gamma \mid \Delta \vdash t : A}{\Gamma \mid \Delta' \vdash t : A}$$ when Δ is a **subsequence** of Δ'

Remarks on the proof

To prove λ-definable $\subseteq$ star-free, we use:

Lemma (in our non-commutative λ-calculus)

For any purely affine A, *the closed terms of type* $A \multimap A$, *modulo* $=_\beta$, *form a* finite and aperiodic monoid.

(No semantics needed: syntax already finite!)

Conclude similarly to Hillebrand & Kanellakis.

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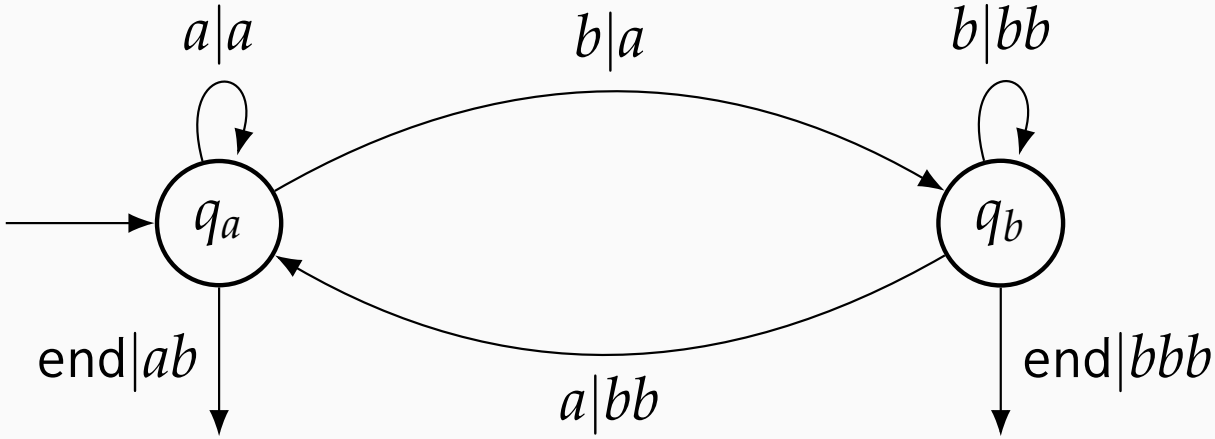
To avoid the scary algebra: a detour through *transducers*.

Sequential transducers: an example



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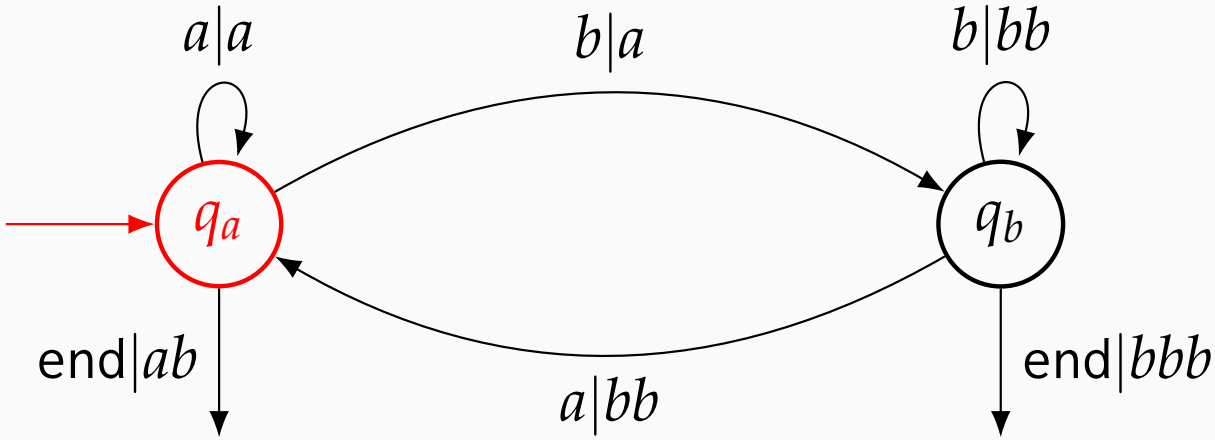

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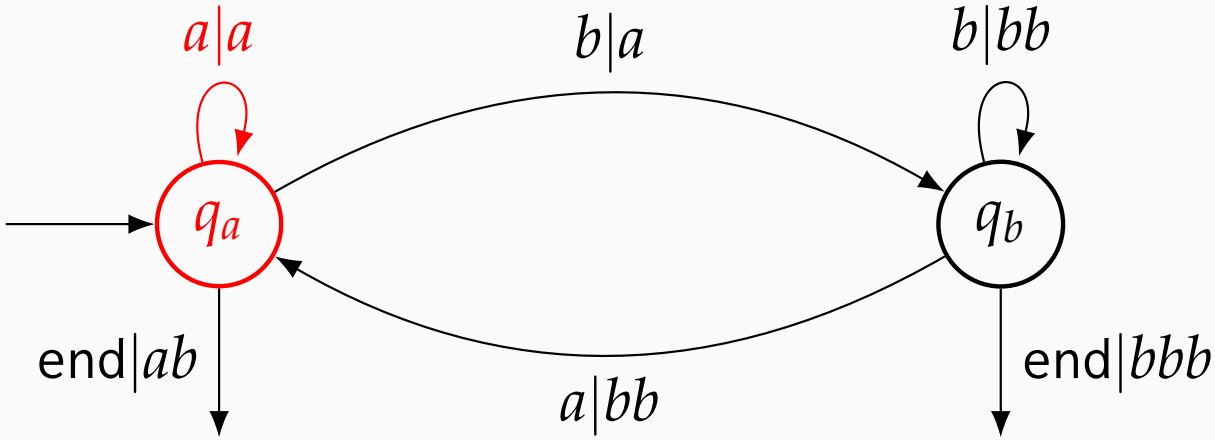

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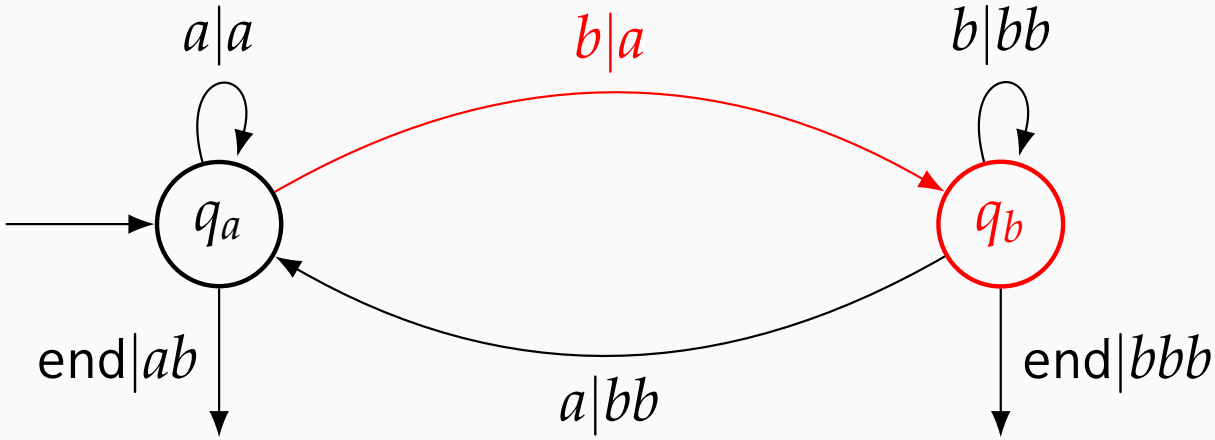

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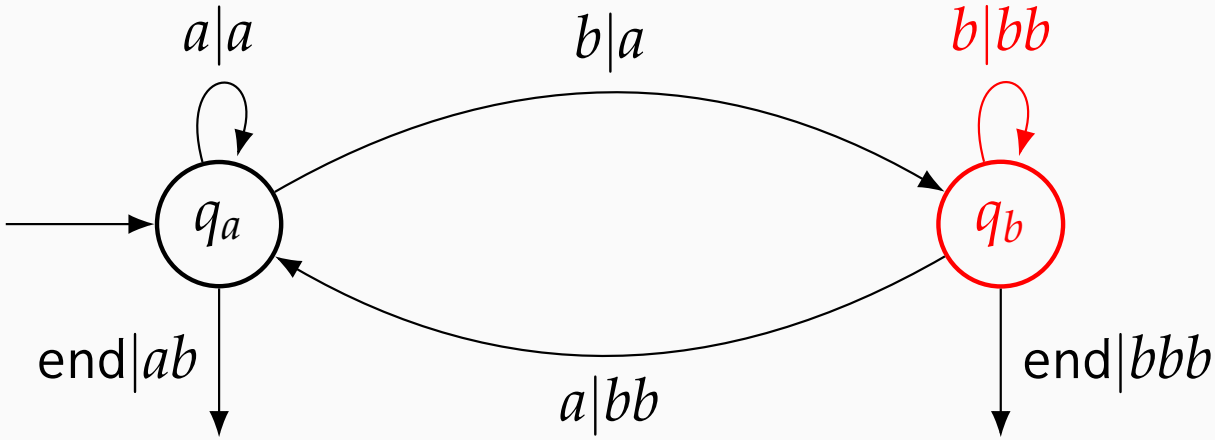

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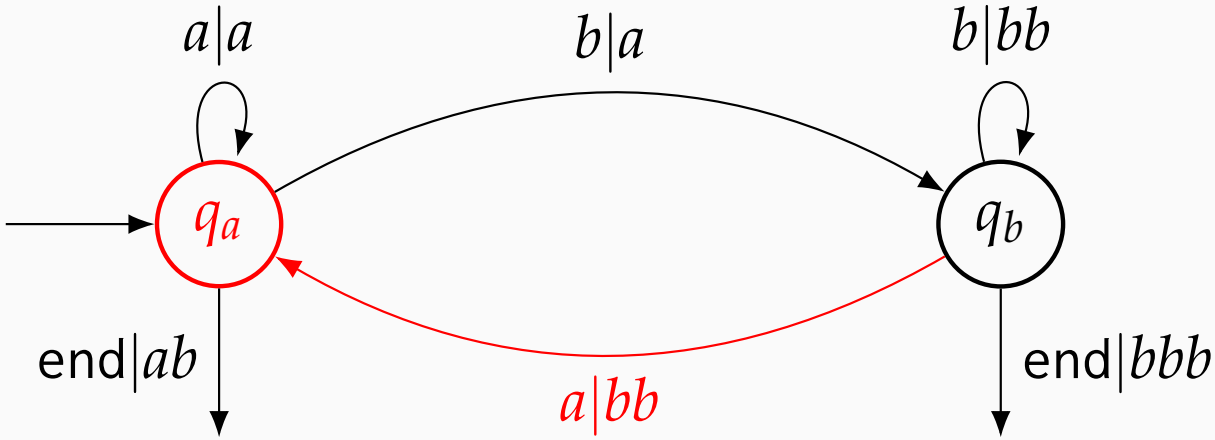

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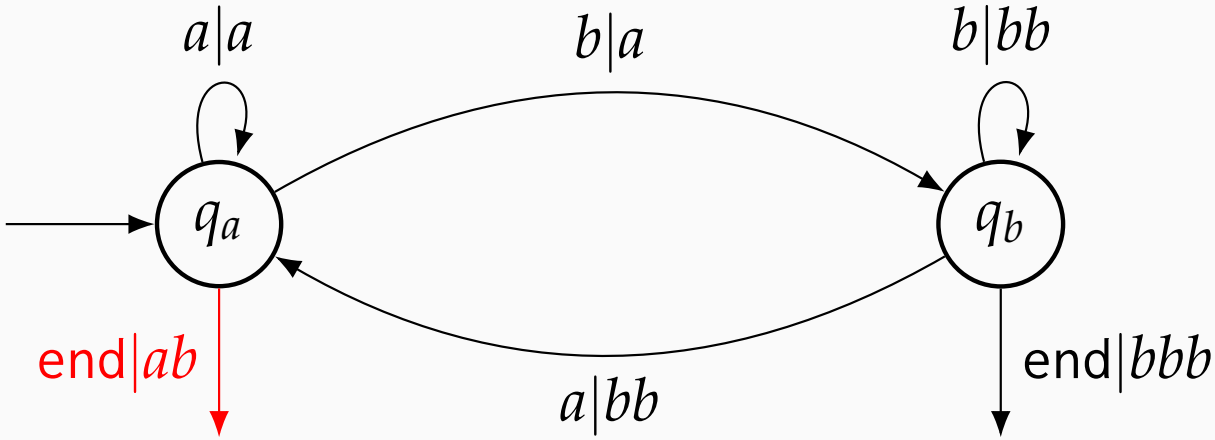

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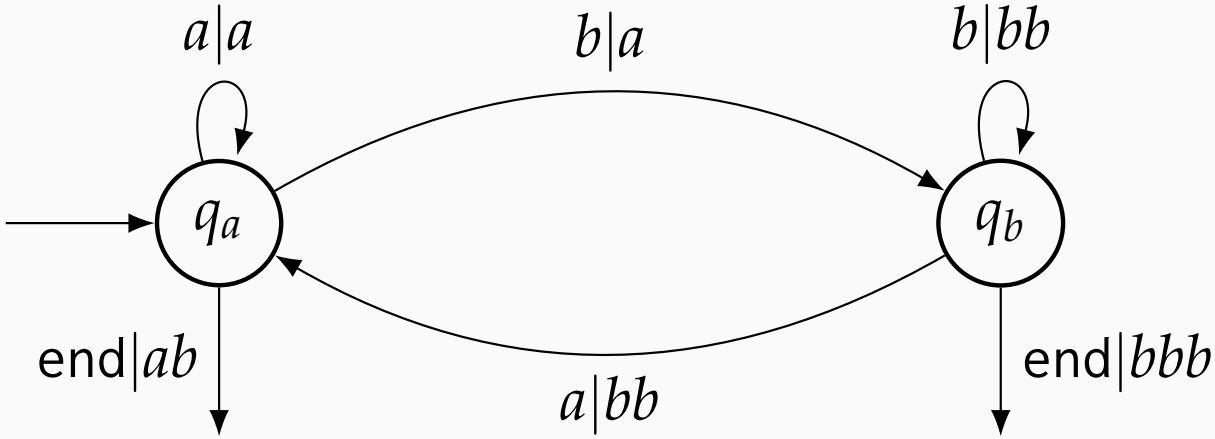

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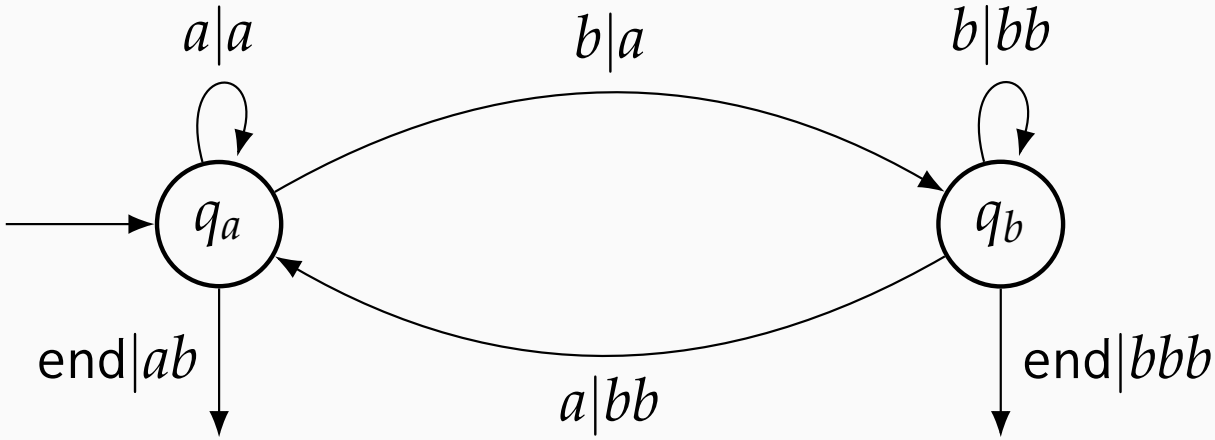

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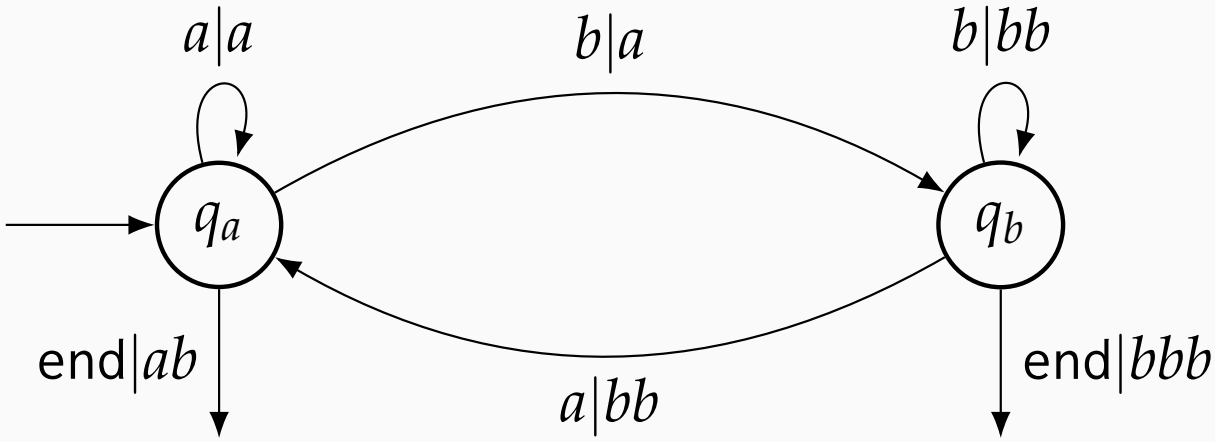
[DeYoung & Pfenning, APLAS'16: this $\iff$ cyclic proofs]

Aperiodic sequential transducers



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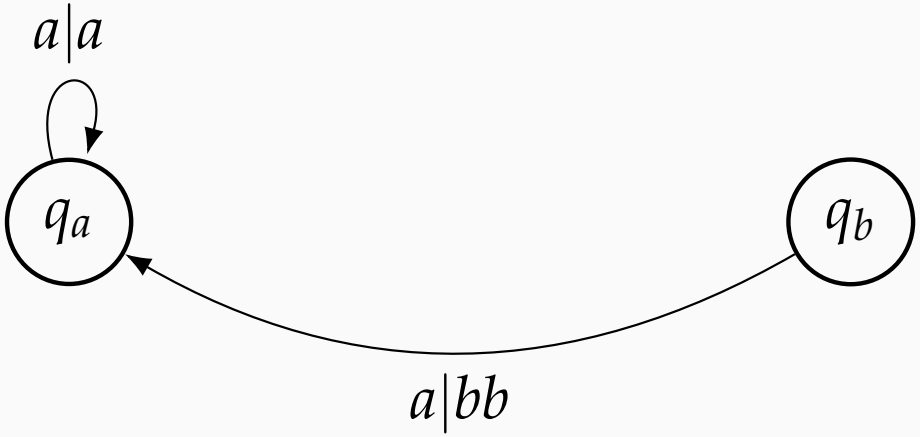

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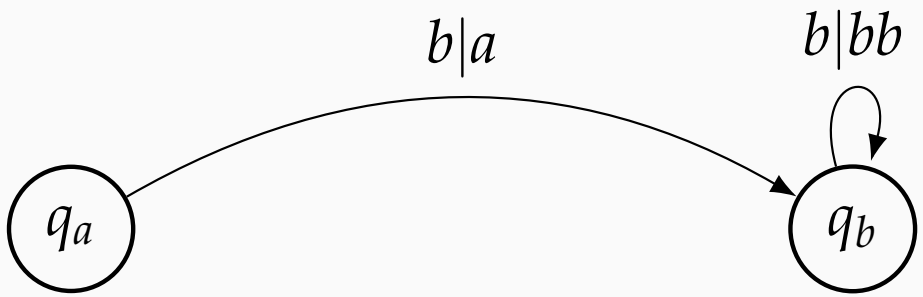

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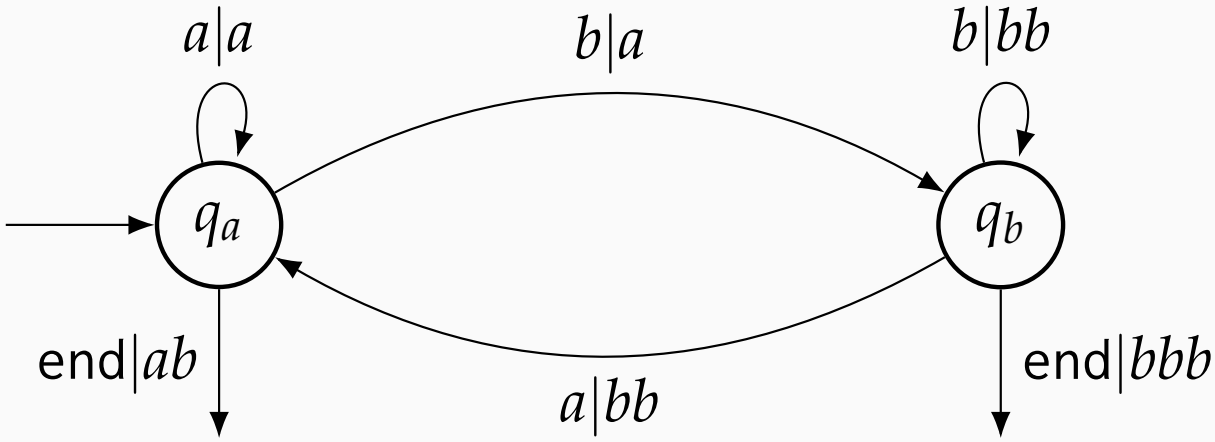

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They generate the *transition monoid* $\{\mathrm{id}, (\_ \mapsto q_a), (\_ \mapsto q_b)\}$

(Remark: this 3-element monoid is the building block in Krohn–Rhodes!)

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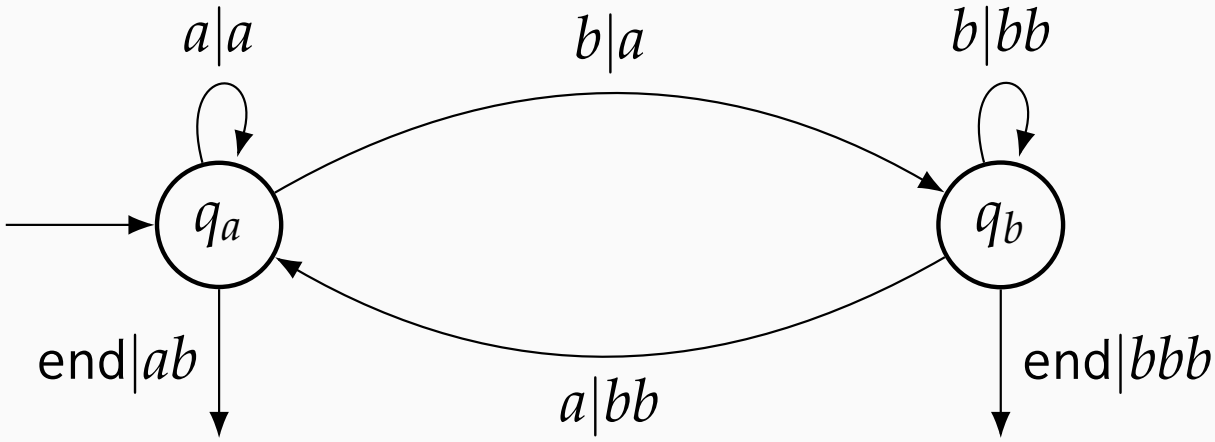

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Lemma

L star-free $\iff$ $L = f^{-1}(\varepsilon)$ for some aperiodic sequential f

The Krohn–Rhodes decomposition, again

Reformulation of the Krohn–Rhodes decomposition

Aperiodic sequential functions are generated by aper. seq. transducers *with 2 states* + usual function composition.

So it's enough to find λ-terms for transducers with 2 states.
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Theorem

Our non-commutative λ-calculus can define all aperiodic sequential functions with terms $t : \mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma[o]$ (A purely affine).

Corollary

It can define all star-free languages with $t : \mathsf{Str}_\Sigma[A] \multimap \mathsf{Bool}[o]$.

String-to-string functions

A byproduct of our proof:

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- $\mathsf{Str}_\Gamma[A] \to \mathsf{Str}_\Sigma[o]$ in simply typed λ-calculus?
  We don't know, even for $\mathsf{Nat}[A] \to \mathsf{Nat}$ ($\Gamma = \Sigma = \{1\}$)
  Negative results by Statman in the 1980s (see also Joly 2001)

Regular functions

Theorem

$\mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma[o]$ (*A purely affine*)
$\iff$ regular (*resp.* aperiodic regular) *functions*
in commutative (*resp. non-commutative*) *affine* λ*-calculus.*

$$\text{sequential fn} \subsetneq \text{rational fn} \subsetneq \textbf{regular fn}$$

Many characterizations; among others:

- *two-way* finite state transducers
- monadic second-order logic (aperiodic case: first-order)
- basic functions + combinators (several variants)
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- *two-way* finite state transducers (2DFT)
  - also, affine λ-calculus $\subseteq$ 2DFT via *geometry of interaction*
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Streaming string transducers

- Finite set of Σ^*-valued *registers* e.g. $R = \{X, Y\}$
- Initial values $R \to \Sigma^*$ e.g. $X_{\text{init}} = Y_{\text{init}} = \varepsilon$
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Execution over *abaa*: start with

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SSTs can also have *states*: their memory is $Q \times (\Sigma^*)^R$ ($|Q| < \infty$)

Equivalent to so-called "HDT0L systems"

Copyless streaming string transducers

Copyless SSTs explained with linear logic connectives:

- Consider the memory as $M = Q \otimes \Sigma^* \otimes \ldots \otimes \Sigma^*$
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Next: linear logic helps us understand automata

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To recover an equality:
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Conclusion (1): summary

$\mathsf{Str}_\Sigma[A] \multimap \mathsf{Bool}[o]$ / $\mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma[o]$ correspond to:

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- Not covered here:
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  - Aperiodic regular fn vs planar geometry of interaction $\longrightarrow$ variant of SSTs using planar diagrams

Conclusion (2): future work

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