Aperiodicity in a non-commutative logic

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Implicit automata in typed λ-calculi: the idea

Suppose the programs of type T in a programming language $\mathcal{P}$ all compute *languages*, something like $T = \texttt{String} \to \texttt{Bool}$.

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My research project w/ Pierre Pradic:
The same thing for **automata theory** instead of complexity

Implicit automata in typed λ-calculi: concretely

Template for a theorem

The languages computed by programs of type T in $\mathcal{P}$ are the same as those decided by the class $\mathcal{A}$ of automata.

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(explaining the λ-calculus and regular languages along the way)

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The λ-calculus

A naive syntactic theory of functions:

$$\begin{array}{rcl} f\,x & \approx & f(x) \\ \lambda x.\ t & \approx & x \mapsto t \\ (\lambda x.\ t)\ u =_\beta t\{x := u\} & \approx & (x \mapsto x^2 + 1)(42) = 42^2 + 1 \end{array}$$

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Church encodings of natural numbers:
morally, $n \in \mathbb{N} \rightsquigarrow \overline{n} : (f, x) \mapsto f^n = f \circ \ldots \circ f(x)$

$$\overline{2} = \lambda f x.\, f\,(f\,x) \qquad :\text{-)}$$

The simply typed λ-calculus

Add a *type system*: specifications for λ-terms

$$t : A \to B \quad \approx \quad \text{“}t \text{ is a function from } A \text{ to } B\text{”}$$

Simple types: built from constant o and binary operation ‘$\to$’

$$\dfrac{f : o \to o \qquad \dfrac{f : o \to o \qquad x : o}{f\,x : o}}{f\,(f\,x) : o}$$

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$$\overline{2} = \lambda f.\ \lambda x.\ f\,(f\,x) : (o \to o) \to o \to o$$

$\mathsf{Nat} = (o \to o) \to o \to o$ is the type of natural numbers

Defining languages in the simply typed λ-calculus

More Church encodings:

Binary strings: think of `fold_right` on a list of characters

$$\texttt{011} \rightsquigarrow (f_0, f_1, x) \mapsto f_0 \circ f_1 \circ f_1(x)$$

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$\texttt{true} : \lambda x.\ \lambda y.\ x \quad \texttt{false} : \lambda x.\ \lambda y.\ y \quad \mathsf{Bool}[A] = A \to A \to A$

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Theorem (Hillebrand & Kanellakis 1996)

This can define all the regular languages*, and only those.*

Regular languages

Many equivalent definitions:

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Theorem (classical)

A language $L \subseteq \Sigma^*$ *is* regular *if and only if there exists a* monoid morphism $\varphi : \Sigma^* \to M$ *to a* finite monoid M *and a subset* $P \subseteq M$ *such that* $L = \varphi^{-1}(P) = \{w \in \Sigma^* \mid \varphi(w) \in P\}$.

Σ: finite alphabet, Σ^*: words over Σ

monoid structure: for $v, w \in \Sigma^*$, $v \cdot w$ = concatenation

morphism: for $w \in \Sigma^*$ with n letters, $\varphi(w) = \varphi(w[0]) \dots \varphi(w[n])$

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- simply typed λ-calculus + Church encodings, as we said!

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Star-free languages and aperiodicity

Star-free languages: regular expressions with *complementation* but *without repetition star* (except Σ^*)

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Definition

A monoid M is *aperiodic* when for any $x \in M$ there exists an exponent $n \in \mathbb{N}$ such that $x^n = x^{n+1}$.

Morally, $(aa)^*$ involves the group $\mathbb{Z}/(2)$: not aperiodic

From aperiodicity to non-commutativity

How to enforce aperiodicity in a typed λ-calculus?
Embedding of $\mathbb{Z}/(2)$ via $\mathsf{not} : \mathsf{Bool} \to \mathsf{Bool}$ $(\mathrm{not}(\mathrm{not}(b)) = b)$

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Idea: make the order of arguments matter

"a function $\lambda b.\ \lambda x.\ \lambda y.\ (\ldots)$ should first use b, then x, then y"
$\longrightarrow$ non-commutative type system

(Lambek 1958: non-commutative λ-calculus for linguistics)

Non-commutative types and linear logic

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(If "*exactly once*": non-commutative *linear* λ-calculus;
correspondence with *planar combinatorial maps* (N. Zeilberger))

Finally, our theorem

Our type system: a base type $o + two$ function arrows:

- non-commutative affine functions: $\lambda^{\circ} x.\ t : A \multimap B$
- unrestricted functions: $\lambda^{\rightarrow} x.\ t : A \to B$

$$\mathsf{Str}_{\{\mathtt{0},\mathtt{1}\}}[A] = (A \multimap A) \to (A \multimap A) \to (A \multimap A)$$

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Theorem (N. & Pradic 2020)

This typed λ-calculus can define all star-free languages*, and only those, with terms of type* $\mathsf{Str}_{\{\mathtt{0},\mathtt{1}\}}[A] \multimap \mathsf{Bool}[o]$ *where A is* purely affine *i.e. does not contain any unrestricted function arrow.*

(*A may vary depending on the language, as in Hillebrand & Kanellakis.*)

With commutative affine types, you'd get regular languages.

Next, some technical details:

- proving (one direction of) Hillebrand–Kanellakis: λ-definable $\subseteq$ regular, by semantic evaluation
- a few words on the proof of our theorem
- rules of our type system

Set-theoretic semantics of the simply typed λ-calculus

Canonical semantics: formalize the idea of λ-terms as functions

- choose set S, $[\![o]\!] = S$, $[\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$
- $t : A \rightsquigarrow [\![t]\!] \in [\![A]\!]$ by induction, e.g. $[\![f\ x]\!] = [\![f]\!]\,([\![x]\!])$
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Lemma

$[\![\overline{w\mathtt{0}}]\!]$ *and* $[\![\overline{w\mathtt{1}}]\!]$ *depend only on* $[\![\overline{w}]\!]$.

Proof: $\mathsf{append0}$ definable so that $\mathsf{append0}\ \overline{w} =_\beta \overline{w\mathtt{0}}$

$[\![\overline{w\mathtt{0}}]\!] = [\![\mathsf{append0}\ \overline{w}]\!] = [\![\mathsf{append0}]\!]\,([\![w]\!])$

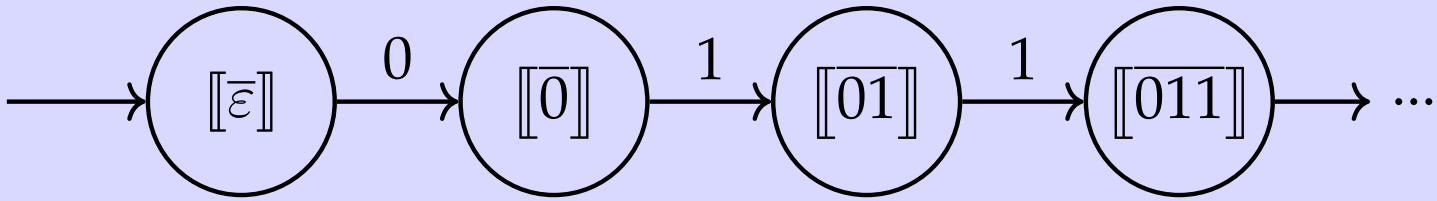
λ-definable languages are regular

Theorem (Hillebrand & Kanellakis, LICS'96)

For any type A and any simply typed λ-term $t : \mathsf{Str}_{\{0,1\}}[A] \to \mathsf{Bool}$, the language $\{w \in \{0,1\}^* \mid t\,\overline{w} =_\beta \mathtt{true}\}$ is regular.

Proof.

We build a DFA with *finite* set of states $Q = [\![\mathsf{Str}_{\{0,1\}}[A]]\!]$ ($\mathrm{Card}(Q)$ depends on A), acceptation as $[\![t]\!](-) = [\![\mathtt{true}]\!]$.



$$t\,\overline{w} =_\beta \mathtt{true} \iff [\![t]\!]([\![\overline{w}]\!]) = [\![\mathtt{true}]\!] \iff w \text{ accepted}$$

(Proof of ($\Leftarrow$): if $\mathrm{Card}([\![o]\!]) \geq 2$ then $[\![\mathtt{true}]\!] \neq [\![\mathtt{false}]\!]$) $\square$

Remarks on the non-commutative case

That was the "hard" inclusion: λ-definable $\subseteq$ regular.

This methods adapts (with a few subtleties) to our case:

Lemma (in our non-commutative λ-calculus)

For any purely affine A, *the closed terms of type* $A \multimap A$, *modulo* $=_\beta$, *form a* finite and aperiodic monoid.

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how do we exploit the aperiodicity assumption?

Theorem (Krohn & Rhodes 1965)

Any aperiodic monoid can be "decomposed" as a wreath product of "building blocks" which are monoids with 3 elements.

Convenient version for us: *aperiodic sequential transducers* = function comp. of transducers w/ *2 states* $\rightsquigarrow \mathsf{Str}_\Gamma[A] \multimap \mathsf{Str}_\Sigma[o]$

A non-commutative affine type system

Typing judgments $\Gamma \mid \Delta \vdash t : A$ for a *set* Γ and an **ordered list** Δ

$$\frac{}{\Gamma \uplus \{x : A\} \mid \varnothing \vdash x : A} \qquad \frac{}{\Gamma \mid x : A \vdash x : A}$$

$$\frac{\Gamma \mid \Delta \vdash t : A \to B \qquad \Gamma \mid \varnothing \vdash u : A}{\Gamma \mid \Delta \vdash t\,u : B} \qquad \frac{\Gamma \uplus \{x : A\} \mid \Delta \vdash t : B}{\Gamma \mid \Delta \vdash \lambda^{\to} x.\, t : A \to B}$$

$$\frac{\Gamma \mid \Delta \vdash t : A \multimap B \qquad \Gamma \mid \Delta' \vdash u : A}{\Gamma \mid \Delta \cdot \Delta' \vdash t\,u : B} \qquad \frac{\Gamma \mid \Delta \cdot (x : A) \vdash t : B}{\Gamma \mid \Delta \vdash \lambda^{\circ} x.\, t : A \multimap B}$$

$\approx$ Polakow & Pfenning's Non-Commutative Intuitionistic Linear Logic

$$\frac{\Gamma \mid \Delta \vdash t : A}{\Gamma \mid \Delta' \vdash t : A}$$ when Δ is a **subsequence** of Δ'

Conclusion / further work

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- in the simply typed λ-calculus: open problem!