Some ideas for a finite geometry of interaction model of second-order MLL

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unfinished work which has been stagnating for a few months

Motivations

Originally: looking for a finite & effective semantics of 2nd-order Multiplicative(-Additive) Linear Logic (M(A)LL2)

- The need arose in implicit complexity, cf. my DICE-FOPARA talks
- More generally: finiteness is important for applications of denotational semantics

Since then, I realized coherence spaces work better.

Also, looking for a parametric/dinatural model, but the model presented here is neither.

Still, it may convey some interesting intuitions and provide a finite dynamic semantics.

(Alternative dynamic semantics: hypercoherences as positional quotient of Blass games + domain-theoretic polymorphism)

A syntactic finite model of MLL2

The idea comes from a finiteness proof for a model of MLL2, obtained by a *quotient of the syntax*.

In propositional MLL (MLL0),
each formula has finitely many cut-free proofs!
i.e. syntactic model for MLL0 is finite

Difficulty of 2nd order case: arbitrarily large $\exists$ witnesses.
Solution: quotient by observational equivalence

- Choose observations which "cannot inspect witnesses"
- Intuition from programming languages: $\exists$ = abstract types, dual to $\forall$ = generic programs

Equivalence for propositional observations

Definition

Let A be a MLL2 formula and $\pi, \pi' : A$. Define $\pi \sim_A \pi'$ as: for any *propositional* MALL formula B, for any proof ρ of $A \vdash B$, $\mathbf{cut}(\pi, \rho)$ and $\mathbf{cut}(\pi', \rho)$ have the same normal form.

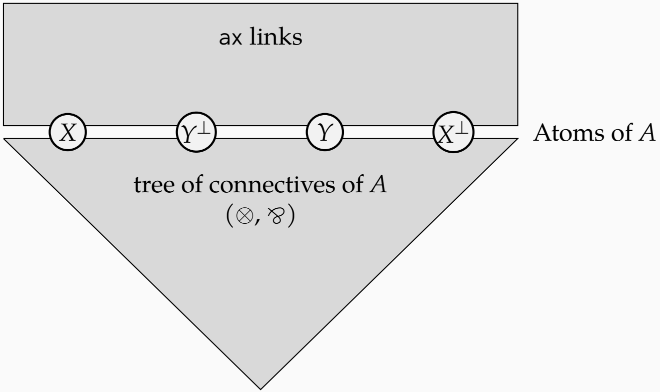
- $\sim$ is a congruence: the quotient is a model of MLL2
- A existential-free $\Rightarrow \sim_A$ trivial
- Example: the proofs of $\exists X.\, X$ cannot be distinguished
  - 2nd-order encodings of units work, e.g. $\top \equiv \exists X.\, X$.

Theorem

For any MLL2 *formula A, there are* finitely many *classes for* $\sim_A$.

The interesting part is the proof, using *proof nets*.

What a propositional MLL proof net looks like



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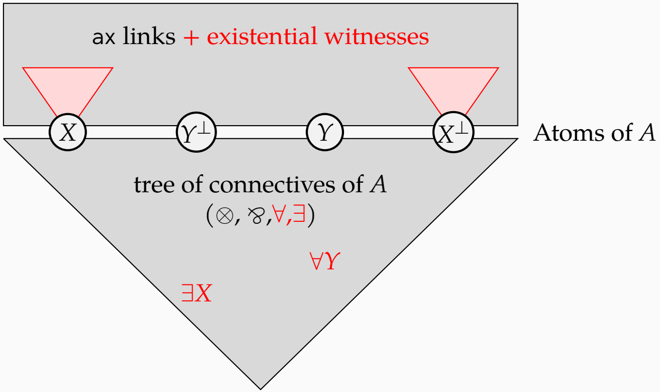
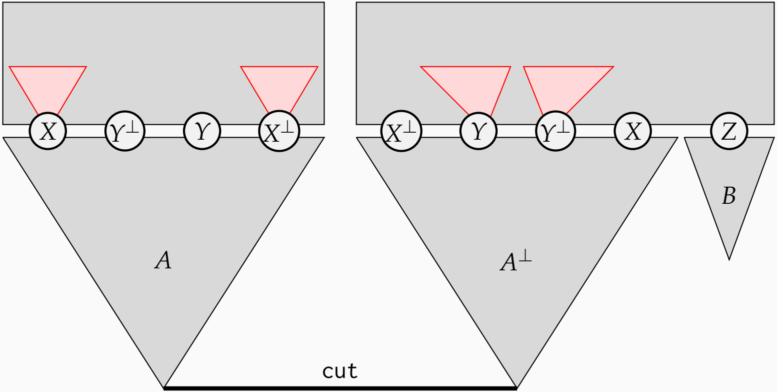


Finiteness theorem (1): proof/observation interaction

A: MLL2 formula; B: propositional MLL formula



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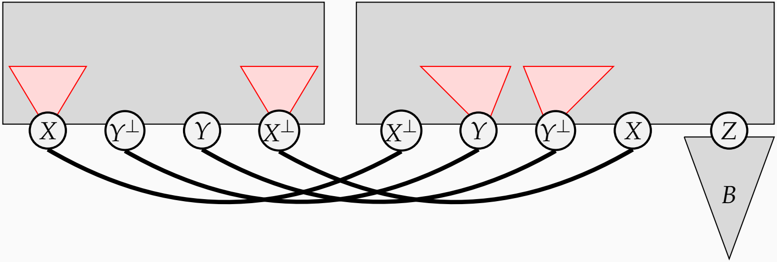
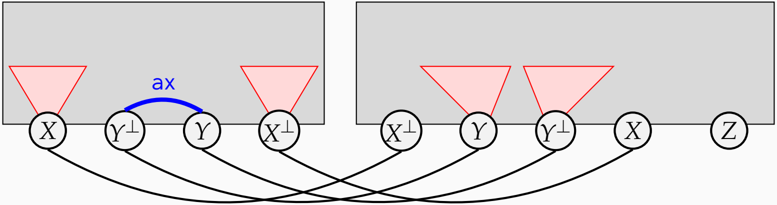


Finiteness theorem (2): eliminating two cuts



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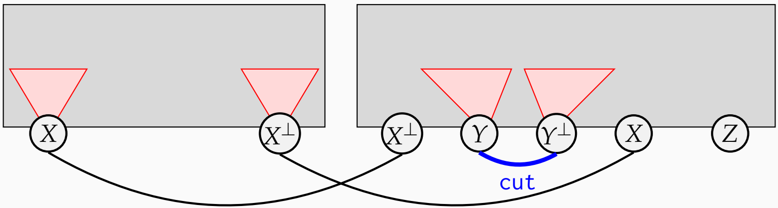

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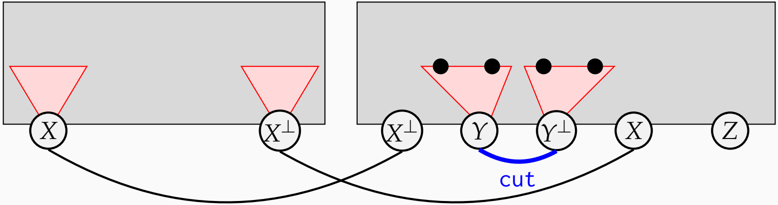

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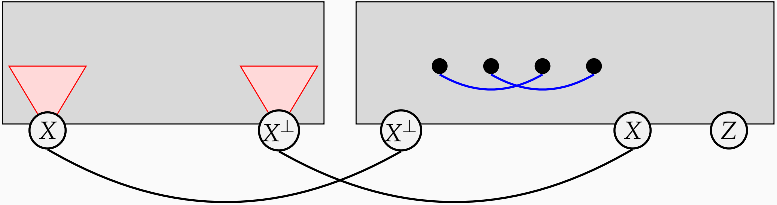

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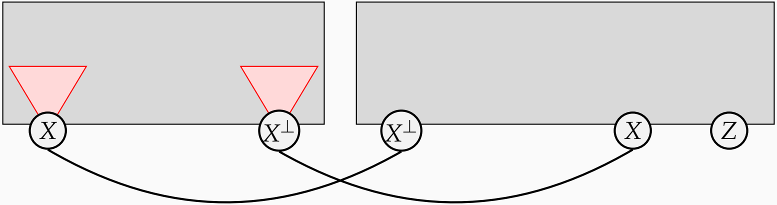
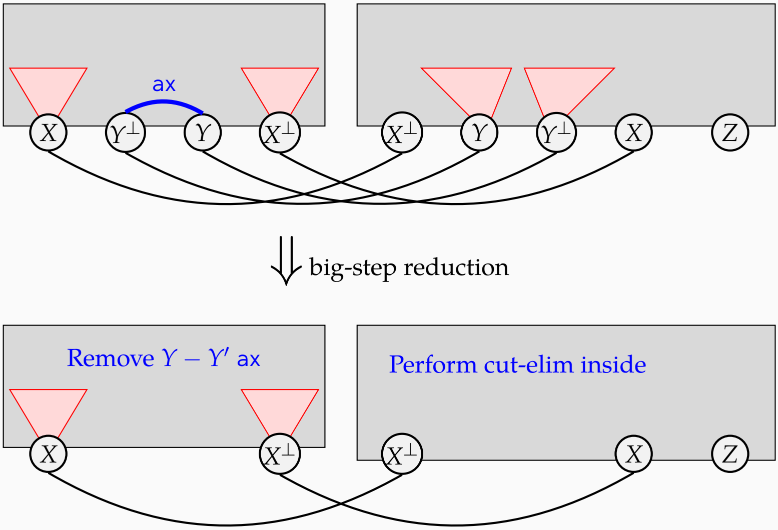
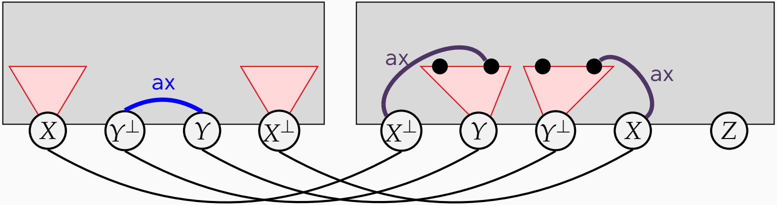


Finiteness theorem (3): big-step reduction



Finiteness theorem (4): normalization

New redexes may appear during reduction.



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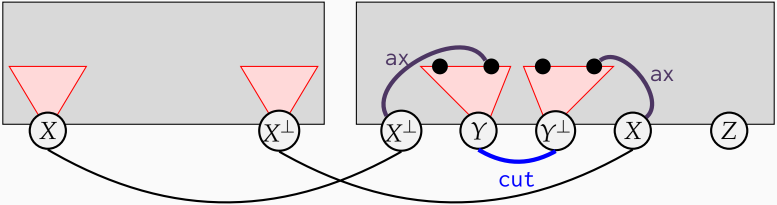

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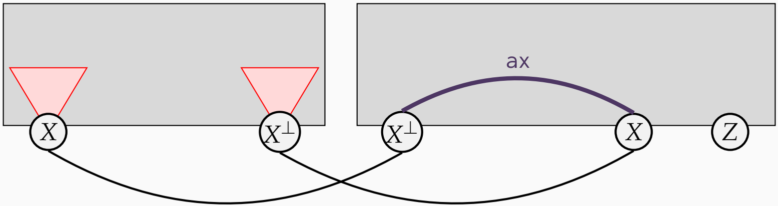

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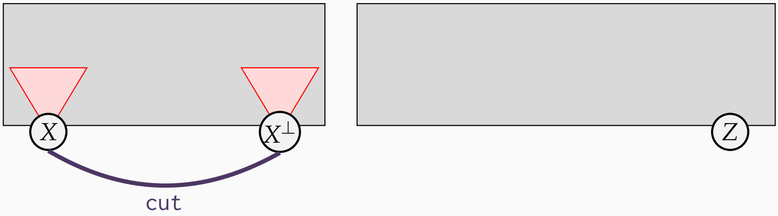

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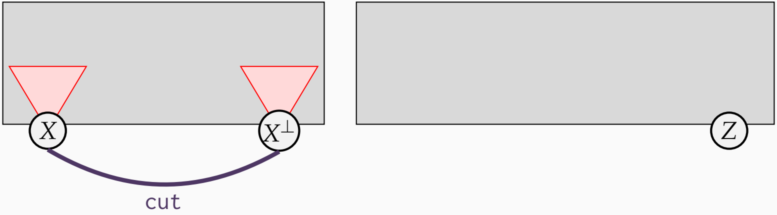

Lemma (Progress)

As long as there remains a `cut`*-link, there is a redex for* $\Rightarrow$.

This is the tricky part of the proof.

MLL2 normalization as a dialogue

One can see this big-step reduction as a *dialogue* between proof and observation, through the interface of the cut formula.

Bound on the number of rounds of the dialogue
$\rightsquigarrow$ bound on number of classes in observational quotient.

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Issue: equivalences classes characterized by an ad-hoc notion of "strategy", *not compositional*.

Can we build an actual denotation model following this idea?
Issue for game semantics: relationship between order of "moves" and structure of the type is unclear.

MLL2 normalization as a fixed point

Let B be a closed MLL2 type.

To a proof of B, we may associate a function $\mathcal{L}_\exists(B) \to \mathcal{L}_\forall(B)$

- $\mathcal{L}_\exists(B)$: partial linkings of $\exists$ variables of B
- $\mathcal{L}_\exists(B)$: partial linkings of $\forall$ variables of B

The "dialogue" then computes a *minimum common fixed point*.
$\longrightarrow$ as in the GoI's *feedback equation*.

A general setting for normalization as feedback:
the Int construction on traced monoidal categories.

$\mathsf{Int}(\mathcal{C})$ is automatically compact closed:
we get propositional MLL for free.

The GoI-like model (1): propositional case

We now work in the category $\mathsf{Int}(\mathcal{C})$ ($\mathcal{C}$ monoidal traced):

- objects: pairs (A, E) of objects of $\mathcal{C}$
- morphisms $(A, E) \to (A', E')$: $\mathcal{C}$-morphisms $A * E' \to E * A'$
- composition $(A, E) \to (A', E') \to (A'', E'')$: take

  $$A * E' \to E * A' \qquad A' * E'' \to E' * A''$$

  feed right E' to left E', left A' to right A',
  and compute minimum fixpoint.
- monoidal product: $(A, E) * (A', E') = (A * A', E * E')$
- duality: $(A, E)^\perp = (E, A)$

Intuition: ask E, answer A.
Will be used with $E = \mathcal{L}_\exists(B)$, $A = \mathcal{L}_\forall(B)$.

The GoI-like model (2): variable types

This is a model of propositional MLL. How to extend to second-order?

General pattern: type with n parameters
$\rightsquigarrow$ functor $F : \mathsf{Int}(\mathcal{C})^n \times (\mathsf{Int}(\mathcal{C})^{\mathrm{op}})^n \to \mathsf{Int}(\mathcal{C})$.

And an element of type F should be a "sufficiently uniform" family of morphisms $1 \to F(X, X)$.

Possible idea: in the case $F(X, X) = K * X^m * (X^\perp)^n$, "uniform" means "routing I/O according to some partial linking"

i.e. morphism $E_K * E^m * A^n \to A_K * A^m * E^n$ sending

- the A on the left (from $X^\perp$) to those on the right (from X)
- the E on the left (from X) to those on the right (from $X^\perp$)

The GoI-like model (3)

The category $\mathcal{C}$ should contain an object $L(m,n)$ for the set of partial linkings on $\{1,\ldots,m\}+\{1,\ldots,n\}$.

Take $\mathcal{C}$ to be e.g. cpo's, or coherence spaces.
Examples of cartesian traced categories $\rightarrow$ "wave-style" GoI.

Informal conjecture:

$$\forall X.\, K * X^m * (X^\perp)^n = K * (L(m,n), 1)$$

$$\exists X.\, K * X^m * (X^\perp)^n = K * (1, L(m,n))$$

$\forall$ adds a part to the answers.
$\exists$ adds a part to the questions.

Note: $\forall X$ and $\exists X$ commute with $(K * -)$ if K contains no X.
As expected, since $* = \otimes = \parr$.

The GoI-like model (4)

How to reflect composition of "sufficiently uniform" families
$K * X^m * (X^\perp)^n \to K' * X^{m'} * (X^\perp)^{n'} \to K'' * X^{m''} * (X^\perp)^{n''}$
as an operation on the $\forall$? Take

$$A{*}E' \to E{*}A'{*}L(n{+}m', m{+}n') \quad A'{*}E'' \to E'{*}A''{*}L(n'{+}m'', m'{+}n'')$$

Feedback matching input/output, we get:

$$A * E'' \to E * A'' * L(n + m', m + n') * L(n' + m'', m' + n'')$$

Need morphism
$L(n + m', m + n') * L(n' + m'', m' + n'') \to L(n + m'', m + n'')$:
should correspond to execution by alternating paths (à la GoI)!
First resolve dialogue of $\exists/\forall$, then propositional normalization.

Inversion of polarities

Suppose $\mathcal{C}$ is cartesian traced.

Types of the form $(A, 1)$ are morally *positive*, they can be duplicated/erased.

Positivation (Hasegawa's exponential): $!(A, E) = (E \to A, 1)$.
Interprets propositional MELL.

$\forall$ seems to be positive, since it acts on the left of the pair!
In particular $\forall X. F$, where the only variable of F is X, is interpreted as positive.
As opposed to usual polarities: $\forall$ inversible, therefore negative.

Morally, $\forall$ "plays" a set of arcs, it is analogous to a $\oplus$;
whereas $\exists$ "reacts", analogously to $\&$.